Joint TZD and related results for Cartesian Product of Real Banach Algebras

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Abstract
Let A and B be two real Banach algebras. We have discussed joint topological zero divisors (JTZD) for Cartesian product of two real Banach algebras and studied the relations between ideals, Carrier space, maximal ideals, and cortex.

Keywords: *Cartesian product of Banach algebra, Joint TZD, Cortex*

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1 INTRODUCTION & PRELIMINARIES

Cartesian product of complex Banach algebras have already been studied [4, 6]. We have mainly discussed JTZD for Cartesian product of two real Banach algebras and studied the relations between ideals, carrier space, maximal ideals and cortex.

Definition 1.1 *Let A and B be Banach algebras (over* P *or* X*) with identities e_A and e_B. The Cartesian product $A\times B$ of A and B is the set $A\times B=\{(a,b):a\in A, b\in B\}$.*

If we define addition, scalar multiplication and multiplication on $A\times B$ coordinate-wise, then it can be easily checked that $A\times B$ is an algebra with identity $e=(e_A,e_B)$. Also $A\times B$ is commutative if and only if A and B both are commutative.

Now, for $(a,b)\in A\times B$, define $\|(a,b)\|=\max\{\|a\|_A,\|b\|_B\}$ where $\|\cdot\|_A$ and $\|\cdot\|_B$ denote the norms in A and B. Then with this norm $A\times B$ is a Banach algebra with identity [6].

If A is a real Banach algebra with identity e_A then $\mathrm{cx}A=\{(a,b):a,b\in A\}$ becomes a commutative complex algebra with identity $(e_A,0)$. Also there exists a norm $\|\cdot\|_{\mathrm{cx}A}$ such that $(\mathrm{cx}A,\|\cdot\|_{\mathrm{cx}A})$ is a Banach algebra [1].

Let G_A, S_A and Z_A denote set of all regular elements, set of all singular elements and set of topological divisors of zero of A. Then it is easy to check that

(i) $G_{A\times B}=G_A\times G_B$,

(ii) $S_{A\times B}=(S_A\times B)\bigcup(A\times S_B)$,

(iii) $Z_{A\times B}=(Z_A\times B)\bigcup(A\times Z_B)$.

Even if $B=A$, the Cartesian product $A\times A$ is different from the complexification cxA. $A\times A$ is a real Banach algebra while cxA is a complex algebra, e.g. $\text{P}^2=\text{P}\times\text{P}$ with coordinatewise operations and maximum norm is a real Banach algebra and X (with usual operations) is the complexification of P . We also note that G_{cxA} and G_A are not related in general, e.g.

(i) Let $A=\text{X}$ and $a=1, b=i$. Then, $a,b\in A$. Also, a and b are invertible in A but $(1,i)$ is not invertible in cxA.

(ii) Let $A=C_{\text{P}}[-1,1]$ and $f(x)=1-x$, $g(x)=1+x$. Then, f and g are not invertible in A. But, (f,g) is invertible in cxA.

If A is a real Banach algebra and $a\in A$, then spectrum of a, denoted by $\sigma(a)$, is defined as $\sigma(a)=\{\lambda\in\text{X}:\lambda=s+it, (a-s)^2+t^2 \textit{ is singular in } A\}$ [1]. We can observe that $\sigma((a,b))=\sigma(a)\bigcup\sigma(b)$.

Theorem 1.2 $\mathrm{cx}A\times\mathrm{cx}B\cong\mathrm{cx}(A\times B)$.

Proof. Define $\phi:\mathrm{cx}A\times\mathrm{cx}B\to\mathrm{cx}(A\times B)$ by $\phi(((a,c),(b,d)))=((a,b),(c,d))$ where $a,c\in A$ and $b,d\in B$. Then ϕ is an isomorphism and a homeomorphism from $\mathrm{cx}A\times\mathrm{cx}B$ to $\mathrm{cx}(A\times B)$.

Because of the above theorem, certain results regarding Cartesian product of real Banach algebras can be obtained using complexification technique. Here we have given proofs, which are valid independent of the scalar field.

We have the following results regarding ideals in Cartesian product of complex Banach algebras.

Remarks 1.3

1. If I_A and I_B are ideals in A and B, then $I_A \times I_B$ is an ideal in $A \times B$.
2. If I be an ideal in $A \times B$, then $P_A(I) = \{a \in A : (a,b) \in I,\ \mathit{for\ some}\ b \in B\}$ an $P_B(I) = \{b \in A : (a,b) \in I,\ \mathit{for\ some}\ a \in A\}$ are ideals in A and B respectively.

Above results for Cartesian product of real Banach algebras can be proved similarly.

If A and B are commutative Banach algebras, then we can talk about their carrier space and maximal ideal space. To study these concepts for the Cartesian product, we introduce some concepts about topology. This will be helpful in understanding the Gelfand topology for the carrier space of the Cartesian product of Banach algebras and that for the factor algebras.

Let X and Y be two topological spaces with topologies τ_X and τ_Y respectively. Then, the union $X \cup Y$ is a topological space with the topology $\tau = \{G \subset X \cup Y : G \cap X \in \tau_X, G \cap Y \in \tau_Y\}$. This topology is known as sum topology on $X \cup Y$. Generally, the spaces X and Y are taken distinct and we say that $X \cup Y$ is a disjoint sum. We will denote it by $X + Y$.

It is proved that if X and Y are compact Hausdorff spaces, then so is $X + Y$ [6]. As we know that correspondence between $Car(A)$ and $\mathcal{M}(A)$ is two to one for a real Banach algebra A, one may think that we will get different result for the Cartesian product for the concepts of carrier space and maximal ideal space [6]. But we get similar results.

Proposition 1.4 $Car(A \times B) = Car(A) + Car(B)$.

Proof. Let $\phi_A \in Car(A)$. We define $\phi : A \times B \to X$ by $\phi((a,b)) = \phi_A(a)$, $(a,b) \in A \times B$. Then, we can easily verify that ϕ is a real homomorphism on $A \times B$, i.e., $\phi \in Car(A \times B)$.

Similarly, for $\phi_B \in Car(B)$ we can define $\phi((a,b)) = \phi_B(b), (a,b) \in A \times B$. Then, $\phi \in Car(A \times B)$. So, $Car(A) + Car(B) \subset Car(A \times B)$.

Conversely, if $\phi \in Car(A \times B)$, then $0 = \phi((0,0)) = \phi((1_A,0)\cdot(0,1_B)) = \phi(1_A,0)\cdot\phi(0,1_B)$. So, either $\phi((1_A,0)) = 0$ or $\phi((0,1_B)) = 0$. We can verify that if $\phi((0,1_B)) = 0$ (resp. $\phi((1_A,0)) = 0$), then $\phi_A(a) = \phi((a,0))$ (resp. $\phi_B(b) = \phi((0,b))$) will be a real homomorphism. So, either $\phi \equiv \phi_A$ or $\phi \equiv \phi_B$. Therefore, $Car(A \times B) \subset Car(A) + Car(B)$.

So, $Car(A \times B) = Car(A) + Car(B)$.

We can also check that the Gelfand topology on $Car(A \times B)$ is the sum of the Gelfand topologies on $Car(A)$ and $Car(B)$. The following corollary is an immediate consequence of Proposition 1.4.

Corollary 1.5 $\mathcal{M}(A \times B) = (\mathcal{M}(A) \times \{B\}) \cup (\{A\} \times \mathcal{M}(B))$.

Proof. Let $M \in \mathcal{M}(A \times B)$. Then, $M = \ker\phi$ for some $\phi \in Car(A \times B)$. As $\phi \in Car(A \times B)$, either $\phi \equiv \phi_A$ or $\phi \equiv \phi_B$.

If $\phi \equiv \phi_A$, then $\ker\phi = \{(a,b) : \phi(a,b) = 0\} = \{(a,b) : \phi_A(a) = 0\}$
$= \{a \in A : \phi_A(a) = 0\} \times B = \ker\phi_A \times B = M_A \times B$. Hence, $\ker\phi = M_A \times B \in \mathcal{M}(A) \times \{B\}$.

Similarly, if $\phi = \phi_B$, then $\ker\phi = A \times M_B \in \{A\} \times \mathcal{M}(B)$. Thus,
$\mathcal{M}(A \times B) \subset (\mathcal{M}(A) \times \{B\}) \cup (\{A\} \times \mathcal{M}(B))$.

Converse is trivial.

2 JOINT TOPOLOGICAL DIVISOR OF ZERO

The concept of JTZD for a complex Banach algebra is studied [7]. We define it for a real Banach algebra and we obtained the relation of JTZD for A and B with that of $A \times B$, where A, B are real Banach algebras.

Definition 2.1 *Let A be a real commutative Banach algebra. A subset S of A is said to be consisting of* ***joint topological zero divisors*** *(JTZD) if for every finite subset $\{x_1,\ldots,x_n\}$ of S,* $d(x_1,\ldots,x_n) = \inf\left\{\sum_{i=1}^{n}\|x_i z\| : z \in A, \|z\| = 1\right\} = 0.$

Equivalently, there exists a net (z_α) in A with $\|z_\alpha\| = 1$ and $\lim_\alpha xz_\alpha = 0$ for each $x \in S$ [2]. In particular, if S is an ideal then it is called an ideal consisting of JTZD. Note that if $S = \{x\}$, then the above definition coincides with topological divisor of zero.

For a complex Banach algebra A, the concept of cortex is studied in [5], where it is defined as a subset of carrier space, $Car(A)$. Here for a real Banach algebra A we define the cortex as a subset of the maximal ideal space $\mathcal{M}(A)$. As far as complex Banach algebra is concerned, $Car(A) \cong \mathcal{M}(A)$ and hence we can also look upon cortex of A as a subset of $\mathcal{M}(A)$.

Definition 2.2 *[3] Let A be a real commutative Banach algebra with identity. The set $\{M \in \mathcal{M}(A) : M \text{consistsofJTZD}\}$ is called the cortex of A and it is denoted by $Cor(A)$.*

Now, we discuss the relation between ideals consisting of JTZD and cortex of $A \times B$ with that of A and B.

Theorem 2.3

1. For $S \subset A$, $T \subset B$ if $S \times T$ consists of JTZD in $A \times B$, then S or T consists of JTZD in A or B respectively.
2. $S \subset A$ (respectively $T \subset B$) consists of JTZD if and only if $S \times B$ (respectively $A \times T$) consists of JTZD in $A \times B$.

Proof. (i) Let $S \times T$ consists of JTZD in $A \times B$ and $x \in S$, $y \in T$. Then, $(x, y) \in S \times T$. So, there exists a net $(z_\alpha)_{\alpha \in \Lambda} = ((x_\alpha, y_\alpha))$ in $A \times B$ with $\|z_\alpha\| = 1$ such that $\lim_\alpha (x, y)(z_\alpha) = 0$. Hence, $\|(x, y)(x_\alpha, y_\alpha)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Therefore, $\|xx_\alpha\| < \varepsilon$ and $\|yy_\alpha\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Now, $1 = \|z_\alpha\| = max\{\|x_\alpha\|, \|y_\alpha\|\}$.

Let $\Lambda_x = \{\alpha \in \Lambda, \|x_\alpha\| = 1\}$, $\Lambda_y = \{\alpha \in \Lambda, \|y_\alpha\| = 1\}$. Then, $\Lambda = \Lambda_x \bigcup \Lambda_y$. Since Λ is directed set it can be checked that atleast one of Λ_x and Λ_y is directed set. If Λ_x is a directed set, then $(x_\alpha)_{\alpha \in \Lambda_x}$ is a net in A with $\|x_\alpha\| = 1$. Also, $\lim_\alpha xx_\alpha = 0$. Hence, S consists of JTZD. Similarly, if Λ_y is a directed set, then T consists of JTZD. Finally, if Λ_x and Λ_y both are directed sets, then S and T both consists of JTZD.

(ii) Let $S \subset A$ consists of JTZD. To show that $S \times B$ consists of JTZD, let $(x, y) \in S \times B$. Then, $x \in S$. Hence, there exists a net (z_α) in A such that $\|z_\alpha\| = 1$ and $\lim_\alpha xz_\alpha = 0$. Hence, $\|xz_\alpha\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. Since, $z_\alpha \in A$ we have $(z_\alpha, 0) \in A \times B$ and $\|(z_\alpha, 0)\| = \|z_\alpha\| = 1$. So, $((z_\alpha, 0))$ is a net in $A \times B$ with $\|(x, y)(z_\alpha, 0)\| = max(\|xz_\alpha\|, 0) = \|xz_\alpha\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, $\lim_\alpha (x, y)(z_\alpha, 0) = 0$. Hence, $S \times B$ consists of JTZD.

Conversely, let $S \times B$ consists of JTZD. To show that S consists of JTZD, let $x \in S$. Then $(x, e_B) \in S \times B$. So there exists a net $((x_\alpha, y_\alpha))$ in $A \times B$ with $\|(x_\alpha, y_\alpha)\| = 1$ and $\|(x_\alpha, y_\alpha)(x, e_B)\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$. So, we have $\|x_\alpha x\| < \varepsilon$ and $\|y_\alpha\| < \varepsilon$ for $\alpha \geq \alpha_\varepsilon$.

Let $\Lambda_x = \{\alpha \in \Lambda : \alpha \geq \alpha_\varepsilon\}$. Then $\|x_\beta\| = 1$ for $\beta \in \Lambda_x$. So, (x_β) is a net with $\|x_\beta\| = 1$ and $\|x_\beta x\| < \varepsilon$. So, $\lim_\beta x_\beta x = 0$. So, S consists of JTZD.

Corollary 2.4 Let I_A and I_B br ideals in A and B respectively.

1. If $I_A \times I_B$ is an ideal consists of JTZD in $A \times B$, then I_A or I_B is an ideal consists of JTZD in A or B respectively.
2. I_A (respectively I_B) consists of JTZD if and only if $I_A \times B$ (respectively $A \times I_B$) consists of JTZD in $A \times B$.

Proof. Results can be proved using remarks 1.3 and Theorem 2.3.

Theorem 2.5 $Cor(A \times B) = (Cor(A) \times \{B\}) \cup (\{A\} \times Cor(B))$.

Proof. Let $M_A \in Cor(A)$. Then, $M_A \times B$ consists of JTZD. Similarly, if $M_B \in Cor(B)$ then $A \times M_B \in Cor(A \times B)$. Therefore, $(Cor(A) \times \{B\}) \cup (\{A\} \times Cor(B)) \subset Cor(A \times B)$.

Conversely, if $M \in Cor(A \times B)$, then by Corollary 1.5, either $M = M_A \times B$ or $M = A \times M_B$ for some maximal ideal M_A of A or M_B of B. If $M = M_A \times B$, then $M_A \times B \subset A \times B$ consists of JTZD. Therefore, $M_A \subset A$ consists of JTZD. Similarly, if $M = A \times M_B$ then $M_B \subset B$ consists of JZD. Therefore, $Cor(A \times B) \subset (Cor(A) \times \{B\}) \bigcup (\{A\} \times Cor(B))$. Hence, $Cor(A \times B) = (Cor(A) \times \{B\}) \bigcup (\{A\} \times Cor(B))$.

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