Existence of best proximity points for cyclic B-contraction mappings

V. Anbukkarasi[a], M. Marudai[a]

[a]Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamil Nadu, India.

Abstract

In this paper, we obtain best proximity points for cyclic B-contraction mappings that unify and extend results of Anthony Eldred et al, Bright et al and many authors.

Keywords: Cyclic B-contraction, Proximity point, Fixed points.

1. Introduction and preliminaries

Anthony Eldred et al[1] introduced the concept of cyclic contraction mappings and obtained best proximity points for cyclic contractions. Bright et al[2] proved fixed point theorems by introducing the new concept B-contraction mapping. In this paper, we introduce the new concept cyclic B-contraction and and obtain best proximity point theorems. Our results unify and extend all the results of Anthony Eldred et al, Bright et al and many authors. The following notions were used subsequently:

$$d(A; B) := \inf\{d(x, y) : x\in A \text{ and } y\in B\}$$

$$A_0 := \{x\in A : d(x, y) = d(A, B) \text{ for some } y\in B\}$$

$$B_0 := \{y\in B : d(x, y) = d(A, B) \text{ for some } x\in A\}$$

A map $T:A\to B$ is said to have proximity point if there exists $x\in A$ such that $d(x,Tx) = d(A,B)$.

Definition 1.1: Let (X,d) be a metric space and let S,T be a selfmaps on X. Then S,T have a common best proximity point if there exists $x\in X$ such that $d(x,Tx) = d(x,Sx) = d(A, B)$.

Definition 1.2 [1]: A subset K of a metric space X is said to be boundedly compact if each bounded sequence in K has a subsequence converging to a point in K.

Suppose X is uniformly convex (and hence reflexive) Banach space with modulus of convexity δ.Then $\delta(\in > 0)$ for $\in>0$ and $\delta(.)$ is strictly increasing. Moreover if $x,y,p \in X$, $R > 0$ and $r\in[0,2R]$,

$$\left.\begin{array}{l}\|x-p\|\le R\\ \|y-p\|\le R\\ \|x-y\|\ge R\end{array}\right\} \Rightarrow \left\|\frac{x+y}{2} - p\right\| \le \left(1-\delta\left(\frac{r}{R}\right)\right)R.$$

Definition 1.3 [1]: Let A,B be nonempty subsets of a metric space X, $T:A\cup B \to A\cup B$ is said to be cyclic contraction if

(i) $T(A)\subseteq B$ and $T(B)\subseteq A$

(ii) $d(Tx,Ty)\le k\, d(x,y)+(1-k)\, d(A,B)$ for some $k\in (0,1)$, for all $x\in A$, $y\in B$.

Definition 1.4: Let $T:X \to X$, where (X,d) is a complete metric space is called a B-contraction if there exists positive real numbers α,β,γ such that $0\le\alpha+2\beta+2\gamma< 1$ for all $x,y\in X$, the following inequality holds:

$$d(Tx,Ty)\le\alpha d(x,y)+\beta[d(x,Tx)+d(y,Ty)]+\gamma[d(x,Ty)+d(y,Tx)] \qquad \text{......................(1)}$$

Lemma 1.5 [1]: Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and $\{x_n\}$,$\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:

(i) $\|z_n-y_n\|\to d(A,B)$

(ii) for every $\in>0$, there exists $n_0 \in N$ such that for all $m>n\geq n_0$,

$\|x_m-y_n\|\leq d(A;B)+\in$.

Then for every $\in>0$, there exists $n_1\in N$ such that for all $m>n\geq n_1$, $\|x_m - z_n\|\leq\in$.

Lemma 1.6[1]: Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and $\{x_n\}$,$\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:

(i) $\|x_n -y_n\|\to d(A,B)$

(ii) $\|z_n - y_n\|\to d(A,B)$.

Then $\|x_n - z_n\|\to 0$.

2. Proximity point theorems for cyclic B-contractions

Definition 2.1: Let A,B be nonempty closed subsets of a metric space X and let $T:A\to B$ is said to be cyclic B-contraction if

(i)T is cyclic and

(ii) there exists nonnegative real numbers numbers α,β,γ such that $\alpha+2\beta+2\gamma< 1$ such that

$d(Tx,Ty)\leq\alpha d(x,y)+\beta[d(x,Tx)+d(y,Ty)]+\gamma[d(x,Ty)+d(y,Tx)] +[1-(\alpha+2\beta+2\gamma)]d(A,B)$

for all $x,y\in A\cup B$ and $\alpha+2\beta+2\gamma<1$.

Proposition 2.2: Let A,B be nonempty subsets of a metric space X. Suppose $T:A\cup B \to A\cup B$ is a cyclic B- contraction map. For any $x_0\in A\cup B$, let $x_{n+1}=Tx_n$ for all $n\in N\cup\{0\}$. Then $d(x_n, x_{n+1}) \to d(A,B)$ as $n \to\infty$.

Proof:

Let $x_0 \in A\cup B$. Then

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1},Tx_n)\\
&\leq \alpha d(x_{n-1},x_n)+\beta[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)]+\gamma[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]\\
&\quad +[1-(\alpha+2\beta+2\gamma)]d(A,B)\\
&\leq \alpha d(x_{n-1},x_n)+\beta[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\gamma[d(x_{n-1},x_{n+1})+d(x_n,x_n)]\\
&\quad +[1-(\alpha+2\beta+2\gamma)]d(A,B)\\
&\leq(\alpha+\beta+\gamma) d(x_{n-1},x_n)+(\beta+\gamma)d(x_n,x_{n+1})+[1-(\alpha+2\beta+2\gamma)]d(A,B).
\end{aligned}$$

Then, we have

$[1- (\beta+\gamma)]d(x_n,x_{n+1})]\leq(\alpha+\beta+\gamma)d(x_{n-1},x_n) + [1- (\alpha+2\beta+2\gamma)]d(A,B)$.

Which gives as

$$d(x_n,x_{n+1})\leq(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]\, d(x_{n-1},x_n)+[1-(\alpha\ 2\beta+2\gamma)] /[1-(\beta+\gamma)]\, d(A,B)$$

$$=(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]\, d(x_{n-1},x_n)+[1-(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]]\, d(A,B)$$

Now,

$$d(x_n,x_{n+1})\leq ((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^2 d(x_{n-2}, x_{n-1}))+([1-(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]])^2 d(A,B).$$

Continuing this process, we get

$$d(x_n,x_{n+1})\leq ((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^n d(x_0, x_1))+([1-(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]])^n d(A,B).$$

Since $\alpha+2\beta+2\gamma<1$, $(\alpha+\beta+\gamma)/[1-(\beta+\gamma)]<1$. Hence $d(x_n,x_{n+1})\to d(A,B)$ as $n\to\infty$.

Corollary 2.1: Let $T : X \to X$, where (X,d) is a complete metric space and T be a B-contraction. Then T has a unique fixed point.

Proof:

The proof of the corollary follows by putting $A = B = X$.

Corollary 2.2: Let A,B be nonempty subsets of a metric space X. Suppose $T : A\cup B \to A\cup B$ is a cyclic contraction map. Then starting with any $x_0 \in A\cup B$, we have $d(x_n,x_{n+1}) \to d(A,B)$ where $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$

Proof:

The proof of the corollary follows by putting $\beta=\gamma=0$.

Proposition 2.3: Let A,B be nonempty closed subsets of a complete metric space X, $T:A\cup B \to A\cup B$ be a cyclic B-contraction map, let $x_0\in A$ and define $x_{n+1}=Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A,then there exists $x \in A$ such that $d(x,T x) = d(A,B)$.

Proof:

Let $\{x_{2nk}\}$ be a subsequence of $\{x_{2n}\}$ and $\lim_{k\to\infty} x_{2nk} = x$ for some $x\in A$.

Now,

$$d(A,B)\le d(x,x2nk-1)\le d(x,x2nk)+d(x2nk,x2nk-1).$$

Taking limit as $n\to\infty$ in the above inequality, we have

$$d(x,x_{2nk}-1) \to d(A,B).$$

Since

$$d(A,B)\le d(x_{2nk},Tx) \le d(x_{2nk}-1,x).$$

As $n\to\infty$, we have $d(x,T x)=d(A,B)$.

Proposition 2.4: Let A,B be nonempty subsets of a metric space X. $T:A\cup B\to A\cup B$ be a cyclic B-contraction map.Then for any $x_0\in A$ and $x_{n+1}=Tx_n$, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.

Proof. Suppose $x_0\in A$(the proof is similar when $x_0\in B$), then by Proposition(2.1),

$d(x_{2n},x_{2n+1}) \to d(A,B)$ as $n\to\infty$. Hence it is enough to prove $\{x_{2n+1}\}$ is bounded. Let $r=((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^2$. Suppose $\{x_{2n+1}\}$ is not bounded. bounded.Then there exists $n_0 \in N$ such that

$$d(x_2,x_{2n0+1})=d(T^2x_0,T^2x_{2n0+1})$$
$$\le r^2d(x_0,x_{2n0+1})+(1-r^2)\, d(A,B).$$

Hence $\dfrac{M-d(A,B)}{r2}$ $+d(A,B)<d(x_0,x_{2n0-1})\le d(x_0,x_2)+d(x_2,x_{2n0-1})\le d(x_0,x_2)+M$.

Thus $M< d(x_0,x_2)/1/1-r^2+ d(A,B)$. Which is a contradiction. Hence $\{x_{2n+1}\}$ is bounded.

Theorem 2.5. Let A,B be nonempty closed subsets of a metric space X and $T:A\cup B\rightarrow A\cup B$ is a cyclic B-contraction. If either A or B is boundedly compact, then there exists $x\in A\cup B$ such that $d(x,Tx) = d(A,B)$.

Proof. It follows directly from propositions (2.3) and (2.4). □

Corollary 2.3. Let A,B be nonempty closed subsets of a metric space X and $T : A\cup B \rightarrow A\cup B$ is a cyclic contraction. If either A or B is boundedly compact, then there exists $x \in A\cup B$ such that $d(x,Tx) = d(A,B)$.

3. Proximity point theorems for cyclic B-contractions on uniformly convex Banach space

Theorem 3.1. Let A,B be nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose $T:A\cup B \rightarrow A\cup B$ is a cyclic B-contraction map, then there exists a unique best proximity point $x \in A$. Further if $x_0 \in A$ and $x_{n+1}=Tx_n$, then $\{x_{2n}\}$converges to the best proximity point.

Proof.

Since $\|x_{2n}-Tx_{2n}\| \rightarrow d(A,B)$ and $\|T^2x_{2n} - Tx_{2n}\| \rightarrow d(A,B)$.By Lemma(1.6), $\|x_{2n} - Tx_{2n}\|\rightarrow 0$. Similarly we can show that $\|Tx_{2n} -Tx_{2(n+1)}\| \rightarrow 0$.We now show that for every $\in > 0$, there exists $n_0 \in N$ such that for all $m > n \geq n_0$, $\|x_{2m} - Tx_{2n}\|\rightarrow d(A,B)+\in$.

Suppose not, then there exists $\in > 0$ such that for all $k\in N$,there exists $m_k >n_k \geq k$ for which

$$\|x_{2m_k}-Tx_{2n_k}\| \geq d(A,B)+\in.$$

$$d(A,B)+\in\leq\|x_{2m_k}-Tx_{2n_k}\|$$
$$\leq\|x_{2m_k} - Tx_{(2m+1)}\| +\|Tx_{(2m+1)}-Tx_{2n_k}\|$$

Hence $\|x_{2m_k} -Tx_{2n_k}\| =d(A,B)+\in$. Now

$$\|x_{2m_k} -Tx_{2n_k}\| \leq\|x_{2m_k} - Tx_{(2m_k+1)}\|+\|Tx_{(2m_k+1)} - Tx_{(2n_k+1)}\|+\|Tx_{(2m_k+1)} -Tx_{2n_k}\|$$

and

$$\|Tx_{(2m_k+1)} - Tx_{(2n_k+1)}\|\leq((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^2 \|x_{2m_k} -Tx_{2n_k}\| +([1-(\alpha+\beta+\gamma)]/[1-(\beta+\gamma)])^2\, d(A,B).$$

Thus

$$d(A,B)+\in \|\leq((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^2\, d(A,B)+\in+([1-(\alpha+\beta+\gamma)]/[1-(\beta+\gamma)])^2\, d(A,B).$$
$$= d(A,B)+ ((\alpha+\beta+\gamma)/[1-(\beta+\gamma)])^2\in$$
$$< d(A,B)+\in$$

which is a contradiction.Therefore $\{x_{2n}\}$ is a Cauchy sequence by Lemma(1.5) andhence converges to some $x\in A$. From proposition(2.3), we have $\|x - Tx\| = d(A,B)$. Now, we have to prove the uniqueness. Suppose x and y are the proximity points for T and $x\neq y$,that is $\|x-Tx\| = d(A,B)$ and $\|y-Ty\| = d(A,B)$ where necessarily, $T^2x = x$ and $T^2y = y$.Therefore

$$\|Tx - y\| = \|Tx -T^2y\| \leq \|x - Ty\|,\ \|Ty - x\| = \|Ty - T^2x\| \leq \|y - Tx\|$$

which implies $\|Ty - x\| = \|y - Tx\|$. But since $\|y - Tx\|>d(A,B)$, it follows that $\|Ty -x\|< \|y - Tx\|$,which is a contradiction. Hence $x = y$.

Corollary 3.1. Let A,B be nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose $T : A \cup B \rightarrow A \cup B$ is a cyclic contraction map, then there exists a unique best proximity point $x\in A$. Further if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best$\{x_{2n}\}$
proximity point.

Proof. The proof of the corollary follows by putting if $\beta=\gamma=0$.

References:

[1] Anthony Eldred, A., Veeramani,: P. Existence and convergence of best proximity points, J. Math. Anal. Appl. 323(2006) 1001-1006

[2] Bright V.S. and Marudai M., Unique fixed point theorems for weakly B-contractive mappings, Far East journal of Mathematical sciences (FJMS), 98(7) 2014, 897-914.

[3] Chatterjea SK., Fixed point theorems, C.R-Acad.Bulgaria Sci.25, 727-730.(1972).

[4] Choudhury B.S. Unique fixed point theorem for weak-C-contractive mappings, Kathmandu Univ. J. Sci. Eng.Technol.5, 6-13(2009).
[5] Kirk W.A., Reich S., Veeramani P., Proximinal retracts and best proximity pair theorems, Numer.Funct.Anal.Optim.24 (2003) 851-862.
[5] Kirk W.A., Srinivasan P.S., Veeramani P., Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4 (2003) 79-89.
[6] Mongholkeha C., Kumam P., Best proximity point theorems for generalized cyclic contractions in ordered metric spaces, J. Optim.Theory. App 155, 215-226(2012).
[7] Shukla D.P. and Shiv Kant Tiwari, Unique fixed point theorem for weakly S-contractive mappings, Gen. Math. Notes, Vol.4, No.1, May 2011, pp.28-34.
[8] Sintunavarat. W, Cho.J, Kumam P: Common fixed point theorems for C-distance in ordered Cone metric space, Comput. Math.Appl. 62, 1969-1978.(2011).