A Study on Compressive Sensing and Reconstruction Approach

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Abstract : This paper gives the conventional approach of reconstructing signals or images from calculated data by following the well-known Shannon sampling theorem. This principle underlies the majority devices of current technology, such as analog-to-digital conversion, medical imaging, or audio and video electronics. The primary objective of this paper is to establish the need of compressive sensing in the field of signal processing and image processing. Compressive sensing (CS) is a novel kind of sampling theory, which predicts that sparse signals and images can be reconstructed from what was in the past thought to be partial information. CS has two distinct major approaches to sparse recovery that each present different benefits and shortcomings. The first, ℓ_1-minimization methods such as Basis Pursuit use a linear optimization problem to recover the signal. This method provides strong guarantees and stability, but relies on Linear Programming, whose methods do not yet have strong polynomially bounded run times. The second approach uses greedy methods that compute the support of the signal iteratively. These methods are much faster than Basis Pursuit, but until recently had not been able to provide the same guarantees. This gap between the two approaches was bridged when we developed and analyzed the greedy algorithm.

Index Terms : ℓ_1-Minimization, Basic Pursuit, Greedy Algorithm, Shannon Sampling Theorem, Sparse, Incoherence.

1. INTRODUCTION

Compressive sensing can be potentially used in various applications which concentrates in the reconstruction of a signal or an image from linear measurements, whilst taking several measurements in particular, an absolute set of measurements is a costly, lengthy, difficult, dangerous, impossible, or otherwise undesired procedure. In addition, there should be a purpose to believe that the signal is sparse in a suitable basis. Empirically, the latter applies to most categories of signals. In computerized tomography, for example, one would like to acquire an image of the inner parts of a human body by taking X-ray images from different angles. Captivating a more or less complete set of images would expose the patient to a hefty and unsafe amount of radiation, so the amount of measurements should be as small as possible, and nevertheless guarantee a better image quality. Such images are usually almost piecewise constant and therefore almost sparse in the gradient, so there is a good reason to consider that compressive sensing is well pertinent. Moreover radar imaging seems to be a very hopeful application of compressive sensing techniques [14]. On is typically monitoring only a small number of targets, so that sparsity is a very realistic assumption. Usual methods for radar imaging also use the sparsity assumption, but only at the very end of the signal processing procedure in order to eliminate the noise in the resulting image. Using sparsity systematically from the very beginning by exploiting compressive sensing methods is consequently a natural approach. The primary numerical experiments are very promising. Further probable applications include wireless communication, astronomical signal and image processing, analog-to-digital conversion, camera design, and imaging. The first naive approach to reconstruct information is by searching the sparsest vector that is consistent with the linear measurements. This leads to the combinatorial ℓ_0-problem, which unfortunately is NP-hard in general. The alternative approach includes ℓ_1– minimization which works with two main properties: null space property (NSP) and restricted isometric property (RIP) [21]. Total-variation minimization, which is closely related to ℓ_1-minimization, is then considered as recovery method. In image processing, the use of total-variation minimization, which is closely connected to ℓ_1-minimization and compressive sensing, first appears in the 1990s in the work of Rudin et al., and was widely applied later on. The use of ℓ_0-minimization and related methods was greatly popularized with the work of Tibshirani on the so-called least absolute shrinkage and selection operator (LASSO). This section introduces the concept of sparsity and the recovery of sparse vectors from incomplete linear and non-adaptive measurements. In particular, an analysis of ℓ_1-minimization as a recovery method is provided. Thenull-space property and the restricted isometry property are introduced and it is shown that they ensure robust sparse recovery. It is actually difficult to show these properties of deterministic matrices and the optimal number m of measurements, and the major breakthrough in compressive sensing results is obtained for random matrices.

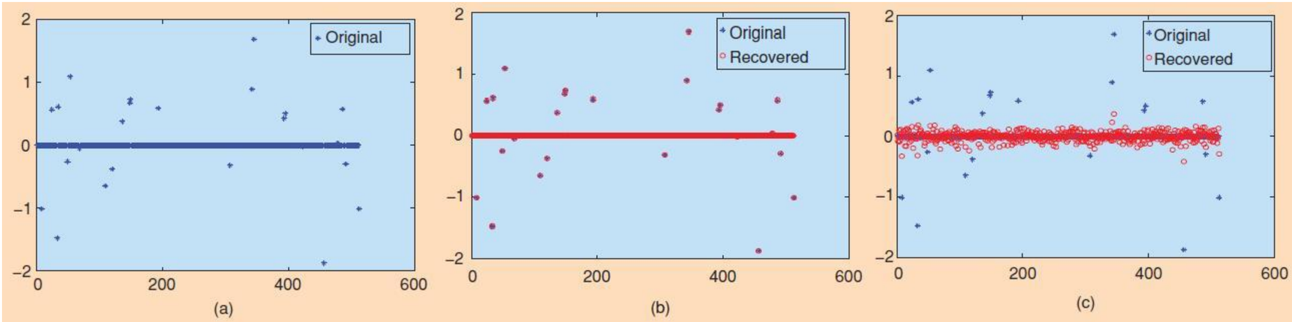

Figure 1: (A) A Sparse Real Valued Signal (B) Its Reconstruction by ℓ1 Minimization. The Reconstruction is Exact. (C) The Reconstruction Obtained by substituting the ℓ1 Norm With The ℓ2 Norm; ℓ1 and ℓ2 gives A Different Answer. The ℓ2 Solution Does Not Provide a Better Approximation to the Original Signal.

2. FUNDAMENTAL PREMISES UNDERLYING CS

Conventional approaches to sampling signals or images follow Shannon's celebrated theorem: the sampling rate must be at least twice the maximum frequency present in the signal (the so-called Nyquist rate). In fact, this principle underlies nearly all signal acquisition protocols used in consumer audio and visual electronics, medical imaging devices, radio receivers, and so on. (For some signals, such as images that are not naturally band limited, the sampling rate is dictated not by the Shannon theorem but by the desired temporal or spatial resolution. However, it is common in such systems to use an antialiasing low-pass filter to band limit the signal Before sampling, and so the Shannon theorem plays an implicit role.) In the field of data conversion, for example, standard analog-to-digital converter (ADC) technology implements the usual quantized Shannon representation: the signal is uniformly sampled at or above the Nyquist rate. The two fundamental premises underlying CS are sparsity, which pertains to the signals of interest, and incoherence, which pertains to the sensing modality. Many natural signals have concise representations when expressed in a convenient basis. Consider, for example, the image in Figure 1(a) and its wavelet transform in (b). Although nearly all the image pixels have nonzero values, the wavelet coefficients offer a concise summary: most coefficients are small, and the relatively few large coefficients capture most of the information. Sparsity articulates the idea that the information rate of a continuous time signal may be much smaller than suggested by its bandwidth, or that a discrete-time signal depends on a number of degrees of freedom which is comparably much smaller than its (finite) length. More precisely, CS exploits the fact that many natural signals are sparse or compressible in the sense that they have concise representations when expressed in the proper basis ψ. Incoherence extends the duality between time and frequency and expresses the idea that objects having asparse representation in ψ must be spread out in the domain in which they are acquired, just as a Dirac or a spike in the time domain is spread out in the frequency domain. Put differently, incoherence says that unlike the signal of interest, the sampling/sensing waveforms have an extremely dense representation in ψ. The crucial observation is that one can design efficient sensing or sampling protocols that capture the useful information content embedded in a sparse signal and condense it into a small amount of data. These protocols are non adaptive and simply require correlating the signal with a small number of fixed waveforms that are

Incoherent with the sparsifying basis. What is most remarkable about these sampling protocols is that they allow a sensor to very efficiently capture the information in a sparse signal without trying to comprehend that signal. Further, there is a way to use numerical optimization to reconstruct the full-length signal from the small amount of collected data. In other words, CS is a very simple and efficient signal acquisition protocol which samples in a signal independent fashion at a low rate and later uses computational power for reconstruction from what appears to be an incomplete set of measurements. Our intent in this article is to overview the basic CS theory that emerged in the works [1][3], present the key mathematical ideas underlying this theory, and survey a

Couple of important results in the field. Our goal is to explain CS as plainly as possible, and so our article is mainly of a tutorial nature. One of the charms of this theory is that it draws from various sub disciplines within the applied mathematical sciences, most notably probability theory. In this review, we have decided to highlight this aspect and especially the fact that randomness can perhaps surprisingly lead to very effective sensing. Mechanisms. We will also discuss significant implications, explain why CS is a concrete protocol for sensing and compressing data simultaneously (thus the name), and conclude our tour by reviewing important applications.

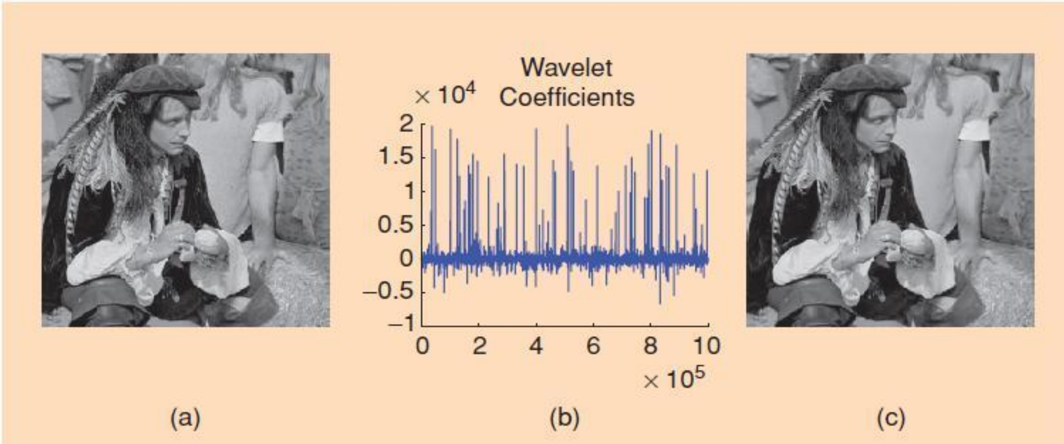

Figure 2: (A) Original Image and (B) Its Wavelet Transform Coefficients (C) The Reconstruction Obtained By Zeroing Out All the Coefficients in the Wavelet Expansion.

3. PROBLEM FORMULATION

Since we will be looking at the reconstruction of sparse vectors, we need a way to quantify the sparsity of a vector. We say that a d-dimensional signal x is s-sparse if it has s or fewer non-zero coordinates:

$$x \in R^d, \|x\|_0 := |supp(x)| \leq s \ll d, \qquad (3.1)$$

Where we note that $\|\cdot\|_0$ is a quasi-norm. For $1 \leq p < \infty$, we denote by $\|\cdot\|p$ the usual p-norm, and $\|x\|\infty = \max |xi|$. In practice, signals are often encountered that are not exactly sparse, but whose coefficients decay rapidly. As mentioned, compressible signals are that satisfying power law decay:

$$|x_i{*}| \leq Ri(-1/q), \qquad (3.2)$$

Where x^* is a non-increasing rearrangement of x, R is some positive constant, and $0 < q < 1$. Note that in particular, sparse signals are compressible.

Sparse recovery algorithms reconstruct sparse signals from a small set of non adaptive linear measurements. Each measurement can be viewed as an inner product with the signal x ∈ Rd and some vector φi ∈ Rd (or in Cd)[1]. If we collect m measurements in this way, we may then consider the m × d measurement matrix Φ whose columns are the vectors φi. We can then view the sparse recovery problem as the recovery of the s-sparse signal x ∈ Rd from its measurement vector u = Φx ∈ Rm. One of the theoretically simplest ways to recover such a vector from its measurements u = Φx is to solve the ℓ_0-minimization problem

$$\min | z\|_0 \quad \text{subject to} \quad \Phi_z = u \quad z \in R^d \quad (3.3)$$

If x is s-sparse and Φ is one-to-one on all 2s-sparse vectors, then the minimize to (3.3) must be the signal x. Indeed, if the minimizer is z, then since x is a feasible solution, z must be s-sparse as well. Since Φz = u, z – x must be in the kernel of Φ. But z − x is 2s-sparse, and since Φ is one-to-one on all such vectors, we must have that z = x. Thus this ℓ_0-minimization problem works perfectly in theory. However, it is not numerically feasible and is NP-Hard in general. Fortunately, work in compressed sensing has provided us numerically feasible alternatives to this NP-Hard problem. One major approach, Basis Pursuit, relaxes the ℓ_0-minimization problem to a ℓ_1-minimization problem. Basis Pursuit requires a condition on the measurement matrix Φ stronger than the simple infectivity on sparse vectors, but many kinds of matrices have been shown to satisfy this condition with number of measurements m = s logO(1) d. The ℓ_1-minimization approach provides uniform guarantees and stability, but relies on methods in Linear Programming [22]. Since there is yet no known strongly polynomial bound, or more importantly, no linear bound on the runtime of such methods, these approaches are often not optimally fast. The other main approach uses greedy algorithms such as Orthogonal Matching Pursuit [9], Stage wise Orthogonal Matching Pursuit, or Iterative Thresholding [16]. Many of these methods calculate the support of the signal iteratively. Most of these approaches work for specific measurement matrices with number of measurements m = O(s log d). Once the support S of the signal has been calculated, the signal x can be reconstructed from its measurements $u = \Phi_x$ as $x = (\Phi_S)\dagger u$, where Φ_S denotes the measurement matrix Φ restricted to the columns indexed by S and † denotes the pseudo inverse. Greedy approaches are fast, both in theory and practice, but have lacked both stability and uniform guarantees [4]. There has thus existed a gap between the approaches. The ℓ_1-minimization methods have provided strong guarantees but have lacked in optimally fast runtimes, while greedy algorithms although fast, have lacked in optimal guarantees. We bridged this gap in the two approaches with our new algorithm Regularized Orthogonal Matching Pursuit (ROMP). ROMP provides similar uniform guarantees and stability results as those of Basis Pursuit, but is an iterative algorithm so also provides the speed of the greedy approach. Our next algorithm, Compressive Sampling Matching Pursuit (CoSaMP) improves upon the results of ROMP, and is the first algorithm in sparse recovery to be provably optimal in every important aspect.

4. ALGORITHMIC APPROACH OF COMPRESSIVE SENSING

Compressed Sensing has provided many methods to solve the sparse recovery problem and thus its applications. There are two major algorithmic approaches to this problem. The first relies on an optimization problem which can be solved using linear programming, while the second approach takes advantage of the speed of greedy algorithms. Both approaches have advantages and disadvantages which are discussed throughout this chapter along with descriptions of the algorithms themselves. First we discuss Basis Pursuit, a method that utilizes a linear program to solve the sparse recovery problem [24].

A. Basic Pursuit

Donoho and his collaborators showed that for certain measurement matrices Φ, this hard problem is equivalent
To its relaxation,

$$\min |z\|_1 \quad \text{subject to} \quad \Phi_z = u \; z \in R^d \quad (4.1)$$

Since the problem (4.1) is not numerically feasible, it is clear that if one is to solve the problem efficiently, a
Different approach is needed. At first glance, one may instead wish to consider the mean square approach, based
On the minimization problem,

$$\min |z\|_2 \text{ subject to } \Phi_z = u \quad z \in R^d \qquad (4.2)$$

Since the minimizer x* must satisfy Φ x* = u = Φ x.
In fact, the minimizer x* to (4.2) is the contact point where
The smallest Euclidean ball centered at the origin meets the subspace K. In this case, the minimizer x* to (4.1) is the contact point where the smallest octahedron centered at the origin meets the subspace K. Since x is sparse, it lies in a low-dimensional coordinate subspace. Thus the octahedron has a wedge at x, which forces the minimizer x* to coincide with x for many subspaces K. Since the ℓ_1-ball works well because of its geometry, one might think to then use an ℓ_p ball for some $0 < p < 1$. The geometry of such a ball would of course lend itself even better to sparsity. The program (4.1) has the advantage over those with $p < 1$ because linear programming can be used to solve it. Basis Pursuit utilizes the geometry of the octahedron to recover the sparse signal x using measurement matrices Φ that satisfies a deterministic property.

B. Greedy Approach

An alternative approach to compressed sensing is the use of greedy algorithms. Greedy algorithms compute the support of the sparse signal x iteratively. Once the support of the signal is compute correctly, the pseudo-inverse of the measurement matrix restricted to the corresponding columns can be used to reconstruct the actual signal x. The clear advantage to this approach is speed, but the approach also presents new challenges [3]. One such greedy algorithm is Orthogonal Matching Pursuit (OMP), put forth by Mallat and his collaborators and analyzed by Gilbert and Tropp [10]. OMP uses sub Gaussian measurement matrices to reconstruct sparse signals [9]. If Φ is such a measurement matrix, then Φ*Φ is in a loose sense close to the identity. Therefore one would expect the largest coordinate of the observation vector y =Φ*Φx to correspond to a non-zero entry of x. Thus one coordinate for the support of the signal x is estimated. Subtracting off that contribution from the observation vector y and repeating eventually yields the entire support of the signal x. OMP is quite fast; both in theory and in practice, but its guarantees are not as strong as those of Basis Pursuit [2].

5. CONCLUSION

Compressed sensing is a new and fast growing field of applied mathematics, signal processing and image processing, which addresses the shortcomings of conventional signal compression. Given a signal with few nonzero coordinates relative to its dimension, compressed sensing seeks to reconstruct the signal from few non adaptive linear measurements. This paper discuss on the two major approaches to the problem emerged, each with its own set of advantages and disadvantages are discussed.

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