

A Study of Solving Linear System of Equations by GAUSS-JORDAN Matrix Method-An Algorithmic Approach

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ABSTRACT

This paper shows how to solve any system of linear equations over an arbitrary field, using the Gauss-Jordan algorithm. For this, we need to define some concepts called row echelon form, elementary row operations, row equivalence and with other knowledge like linear equation, matrices and determinants also. By knowing all the above information here, we designed the Gauss-Jordan Algorithm to solve linear equations with some examples this will be useful to write the computational program to solve the system of equations using Gauss-Jordan method.

KEYWORDS

Row-echelon form, reduced row-echelon form, Elementary row operations, Row equivalence, augmented matrix, Gauss-Jordan Algorithm.

INTRODUCTION

The Gauss-Jordan method is one of the methods to solve system of linear equations using matrix method. Now a days these methods are incorporated in the computational field to solve these type of equations using the program code to compute the solutions. Therefore, by knowing the basic knowledge about the process one can easily write the computational codes in the relevant computer language.

GAUSS-JORDAN Method of solving linear equations.

We show how to solve any system of linear equations over an arbitrary field, using the GAUSS-JORDAN algorithm. We first need to define some terms.

Definition (Row-echelon form) A matrix is in row-echelon form if

- a. All zero rows (if any) are at the bottom of the matrix and
- b. If two successive rows are non-zero, the second row starts with more zeros than the first (moving from left to right).

For example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in row-echelon form,}$$

Whereas the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is not in row-echelon form.}$$

The zero matrix of any size is always in row-echelon form.

Definition (Reduced row-echelon form)

A matrix is in reduced row-echelon form if

1. It is in row-echelon form,
2. The leading (leftmost non-zero) entry in each non-zero row is 1,
3. All other elements of the column in which the leading entry 1 occurs are zeros.

For example the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ are in reduced row-echelon form, whereas the matrices}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not in reduced row-echelon form, but are in row-echelon form.

The zero matrix of any size is always in reduced row-echelon form.

Notation: If a matrix is in reduced row-echelon form, it is useful to denote the column numbers in which the leading entries 1 occur, by c_1, c_2, \dots, c_r , with the remaining column numbers being denoted by c_{r+1}, \dots, c_n , where r is the number of non-zero rows.

For example, in the 4 x 6 matrix above, we have $r = 3, c_1=2, c_2=4, c_3=5, c_4=1, c_5=3, c_6=6$.

The following operations are the ones used on systems of linear equations and do not change the solutions.

Definition (Elementary row operations) There are three types of elementary row operations that can be performed on matrices.

1. Interchanging two rows:
 $R_i \leftrightarrow R_j$ interchanges rows i and j .
2. Multiplying a row by a non-zero scalar:

$R_i \rightarrow tR_i$ multiplies row i by the non-zero scalar t .

3. Adding a multiple of one row to another row:

$R_j \rightarrow R_j + tR_i$ adds t times row i to row j .

Definition (Row equivalence) Matrix A is row-equivalent to matrix B , if B is obtained from A by a sequence of elementary row operations.

Example. Working from left to right.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 + 2R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} \quad R_1 \rightarrow 2R_1 \quad \begin{bmatrix} 2 & 4 & 0 \\ 1 & -1 & 2 \\ 4 & -1 & 5 \end{bmatrix} = B$$

Thus A is row-equivalent to B . clearly B is also row-equivalent to A , by performing the inverse row-operations $R_1 \rightarrow \frac{1}{2} R_1$, $R_2 \leftrightarrow R_3$, $R_2 \rightarrow R_2 - 2R_3$ on B .

It is not difficult to prove that if A and B are row-equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets – a solution of the one system is a solution of the other.

For example, the systems whose augmented matrices are A and B in the above example are respectively

$$\begin{cases} x + 2y = 0 \\ 2x + y = 1 \\ x - y = 2 \end{cases} \quad \text{and} \quad \begin{cases} 2x + 4y = 0 \\ x - y = 2 \\ 4x - y = 5 \end{cases} \quad \text{and these systems have precisely the same solutions.}$$

The Gauss-Jordan algorithm

We now describe the GAUSS-JORDAN ALGORITHM. This is a process, which starts with a given matrix A and produces a matrix B in reduced row-echelon form, which is row-equivalent to A . If A is the augmented matrix of a system of linear equations, then B will be a much simpler matrix than A from which the consistency or inconsistency of the corresponding system is immediately apparent and in fact, the complete solution of the system can be follow.

STEP 1

Find the first non-zero column moving from left to right, (column c_1) and select a non-zero entry from this column. By interchanging rows, if necessary, ensure that the first entry in this column is non-zero. Multiply row 1 by the multiplicative inverse of a_{1c_1} thereby converting a_{1c_1} to 1. For each non-zero element a_{ic_1} , $i > 1$, (if any) in column c_1 , add $-a_{ic_1}$ times row 1 to row i , thereby ensuring that all elements in column c_1 , apart from the first, are zero.

STEP 2

If the matrix obtained at Step 1 has its 2nd, . . . , mth rows all zero, the matrix is in reduced row-echelon form. Otherwise, suppose that the first column, which has a non-zero element in the rows below the first, is column c_2 . Then $c_1 < c_2$. By interchanging rows below the first, if necessary, ensure that a_{2c_2} is non-zero. Then convert a_{2c_2} to 1 and by adding suitable multiples of row 2 to the remaining rows, where necessary, ensure that all remaining elements in column c_2 are zero.

The process is repeated and will eventually stop after r steps, either because we run out of rows, or because we run out of non-zero columns. In general, the final matrix will be in reduced row-echelon form and will have r non-zero rows, with leading entries 1 in columns c_1, \dots, c_r , respectively.

Example.

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 2 & -2 & 5 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_1 \rightarrow \frac{1}{2}R_1 \quad R_3 \rightarrow R_3 - 5R_1 \quad \begin{bmatrix} 1 & 1 & -1 & 5/2 \\ 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \end{bmatrix}$$

$$R_2 \rightarrow \frac{1}{4}R_2 \quad \begin{cases} R_1 \rightarrow R_1 + R_2 \\ R_3 \rightarrow R_3 - 4R_2 \end{cases} \quad \begin{bmatrix} 1 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -15/2 \end{bmatrix}$$

$$R_3 \rightarrow \frac{-2}{15}R_3 \quad R_1 \rightarrow R_1 - \frac{5}{2}R_3 \quad \begin{bmatrix} 1 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row-echelon form.

REMARK: It is possible to show that a given matrix over an arbitrary field is row-equivalent to precisely one matrix, which is in reduced row-echelon form.

A flow-chart for the Gauss-Jordan algorithm, is presented in figure below.

Systematic solution of linear systems

Suppose a system of m linear equations in n unknowns x_1, \dots, x_n has augmented matrix A and that A is row-equivalent to a matrix B which is in reduced row-echelon form, via the Gauss-Jordan algorithm. Then A and B are $m \times (n+1)$. Suppose that B has r non-zero rows and that the leading entry 1 in row i occurs in column number c_i , for $1 \leq i \leq r$. Then $1 \leq c_1 < c_2 < \dots, < c_r \leq n+1$

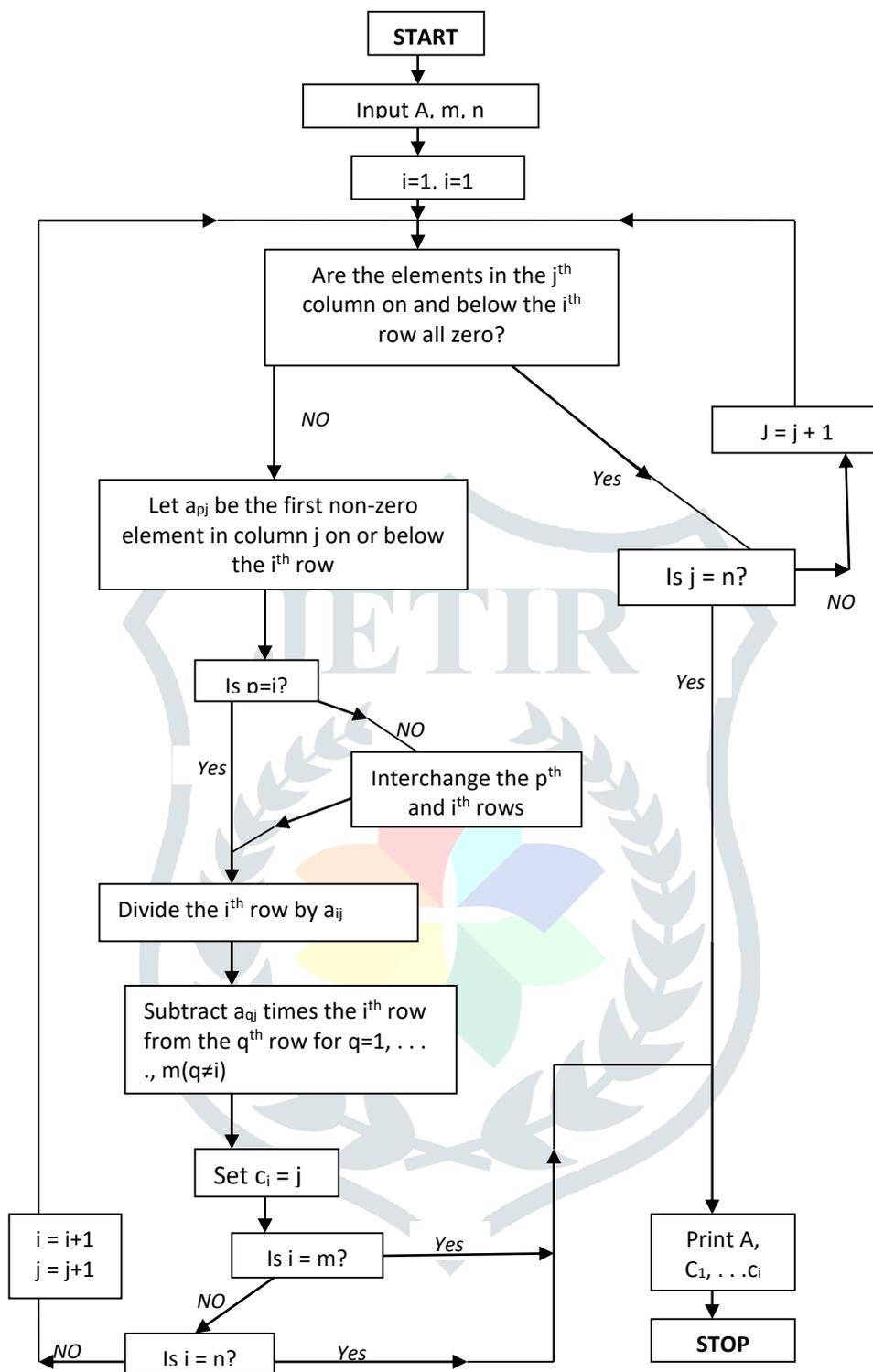


Figure 1: Diagram showing Gauss-Jordan Algorithm

Also assume that the remaining column numbers are c_{r+1}, \dots, c_{n+1} ,
 Where $1 \leq c_{r+1} < c_{r+2} < \dots < c_n \leq n+1$

Case 1: $c_r = n+1$. The system is inconsistent. For the last non-zero row of B $[0, 0, \dots, 1]$ and the corresponding equation is $0x_1 + 0x_2 + \dots + 0x_n = 1$.

Which has no solutions. Consequently, the original system has no solutions.

Case 2: $c_r \leq n$. The system of equations corresponding to the non-zero rows of B is inconsistent. First, notice that $r \leq n$ here. If $r = n$, then $c_1 = 1, c_2 = 2, \dots, c_n = n$ and

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 & d_1 \\ 0 & 1 & \dots & 0 & d_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & d_n \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

There is a unique solution $x_1 = d_1, x_2 = d_2, \dots, x_n = d_n$.

If $r < n$, there will be more than one solution (infinitely many if the field is infinite). For all solutions are obtained by taking the unknowns x_{c_1}, \dots, x_{c_r} as dependent unknowns and using the r equations corresponding to the non-zero rows of B to express these unknowns in terms of the remaining independent unknowns $x_{c_{r+1}}, \dots, x_{c_n}$, which can take on arbitrary values:

$$\begin{aligned} x_{c_1} &= b_{1c_{r+1}} x_{c_{r+1}} + \dots + b_{1c_n} x_{c_n} \\ &\vdots \\ x_{c_r} &= b_{rc_{r+1}} x_{c_{r+1}} + \dots + b_{rc_n} x_{c_n}. \end{aligned}$$

In particular, taking $x_{c_{r+1}} = 0, \dots, x_{c_{n-1}} = 0$ and $x_{c_n} = 0, 1$ respectively, produces at least two solutions.

Theoretical Examples to solve this type of system of equations

1. Example. Solve the system $x + y = 0; x - y = 1; 4x + 2y = 1$

Solution: The augmented matrix of the system is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \text{ which is row equivalent to } B = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We read off the unique solution $x = \frac{1}{2}, y = -\frac{1}{2}$

(Here $n = 2, r = 2, c_1 = 1, c_2 = 2$. Also $c_r = c_2 = 2 < 3 = n+1$ and $r = n$)

2. Example. Solve the system $2x_1 + 2x_2 - 2x_3 = 5; 7x_1 + 7x_2 + x_3 = 10; 5x_1 + 5x_2 - x_3 = 5$.

Solution: The augmented matrix is

$$A = \begin{bmatrix} 2 & 2 & -2 & 5 \\ 7 & 7 & 1 & 10 \\ 5 & 5 & -1 & 5 \end{bmatrix} \text{ which is row equivalent to } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We read off inconsistency for the original system.

(Here $n = 3, r = 3, c_1 = 1, c_2 = 3$. Also $c_r = c_3 = 4 = n+1$)

3. Example. Solve the system $x_1 - x_2 + x_3 = 1; x_1 + x_2 - x_3 = 2$.

Solution: The augmented matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix} \text{ which is row equivalent to } B = \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & -1 & 1/2 \end{bmatrix}$$

The complete solution is $x_1 = 3/2$, $x_2 = 1/2 + x_3$, with x_3 arbitrary.

(Here $n = 3$, $r = 2$, $c_1 = 1$, $c_2 = 2$. Also $c_r = c_2 = 2 < 4 = n+1$ and $r < n$).

- 4. Example.** Solve the system $6x_3 + 2x_4 - 4x_5 - 8x_6 = 8$; $3x_3 + x_4 - 2x_5 - 4x_6 = 4$; $2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2$; $6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1$.

Solution: The augmented matrix is

$$A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{bmatrix} \text{ which is row equivalent to}$$

$$B = \begin{bmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 & 1/24 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 & 5/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The complete solution is, $x_1 = \frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5$, $x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5$, $x_6 = \frac{1}{4}$, with x_2, x_4, x_5 arbitrary. (Here $n = 6$, $r = 3$, $c_1 = 1$, $c_2 = 3$, $c_3 = 6$; $c_r = c_3 = 6 < 7 = n+1$; $r < n$)

CONCLUSION

Here by making use of this, Gauss-Jordan algorithm one can easily write the program code in different platforms to get desired solutions for the system of linear equations. This paper allows the program writers in different computer languages in easy and quick glimpse of idea regarding this type problem solving. Therefore, the Gauss-Jordan Algorithm to solve linear equations with some examples this will be useful to write the computational program to solve the system of equations using Gauss-Jordan method.

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