An Algorithm for Multi-level Fractional Programming Problem Using Goal Programming

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Abstract

This paper considers a multilevel programming problem in which the objective functions are linear fractional and the constraints are linear. An algorithm based on pre-emptive goal programming approach is developed to obtain an optimal feasible solution. The higher level decision makers (DMs) provides the preferred values of the decision variables under their control and the target value of their objective functions to the next level DM to formulate a goal programming problem equivalent to the given multi-level programming problem. The proposed method is illustrated with the help of a tri-level programming problem.

Keywords: Fractional programming, multi-level programming, goal programming, aspirational levels.

1. Introduction

Multi-level programming problems (MLPP) are the characterization of mathematical programming to solve decentralized planning problems with multiple decision makers (DMs) where each unit seeks its own interest.

Multi-level programming problem is a sequence of optimization problems in which the constraints region of one is determined implicitly by the solution of the other levels. Such problems can be formulated as a series of nested linear mathematical programs to be solved simultaneously over a single poly-hedral region. In MLPP different DMs located at different hierarchical levels, each independently controls a set of decision variables. The highest level DM (HLDM) makes his decision in full view of the lower level DM (LLDM). Each DM attempts to optimize its objective function and is affected by the action of the other DM.

The general MLPP is

(P1) $$\max_{X_1} f_1(X) = c_{11}X_1 + c_{12}X_2 + \cdots + c_{1k}X_k$$

$$\max_{X_2} f_2(X) = c_{21}X_1 + c_{22}X_2 + \cdots + c_{2k}X_k$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\underset{X_k}{\mathrm{Max}}\, f_k(X) = c_{k1}X_1 + c_{k2}X_2 + \cdots + c_{kk}X_k$$

subject to

$$A_{i1}X_1 + A_{i2}X_2 + \cdots + A_{ik}X_k = b_i, \quad i = 1, 2, ..., m$$

$$X_1, X_2, ..., X_k \geq 0$$

with one DM at each level, n decision variables and m constraints. Let $X = (X_1, X_2, ..., X_k)$, $n = n_1 + n_2 + \cdots + n_k$, where the decision vector X_k is under the control of the k^{th} level DM and has n_k number of decision variables

Multi-level programming is particularly appropriate for problems with the following characteristics.

- There exists interacting decision making within a predominantly hierarchical structure.
- Each lower level executes its policies after and in view of the decisions of the higher level.
- Each decision making unit optimizes its own function independently of the other units but is affected by the actions of other units as an externality.
- The external effect on a DMs problem is reflected both in his objective function and his set of feasible decisions.

This decision making process is extremely practical to such organization structural levels as agriculture, government policy, economic systems etc. and is especially suitable for conflict resolution.

The Stackelberg strategy has been employed as a solution concept when decision problems are modeled as MLP problems. Computation methods for the Stackelberg solution are classified roughly into three categories: the vertex enumeration approach [6] based on the characterstic

that an extreme point of a set of best responses of the DM at the lower level is also that of a set of common constraints, the Kuhn-Tucker approach in which the upper levels with constraints including lower levels problem is solved and the penalty function approach which adds a penalty term to the upper level's objective function so as to satisfy optimality of the lower level's problem.

Bard [2] and Wen and Bialas [13] have proposed an algorithm for tri-level programming problems. Bard [2] formulates a normal nonlinear programming problem by using the Kuhn-

Tucker conditions for the problems of the third level and the second level and proposes a cutting plane algorithm employing a vertex search procedure to solve a tri-level linear programming problem. Wen and Bialas [13] developed a hybrid algorithm to solve a tri-level linear programming problem. The algorithm adopts the k^{th} best extreme point and the complementary pivot algorithm to check feasibility.

In many practical applications like cutting stock problems, ore blending problems, optimal policy for Markovian chains, sensitivity of linear programming problems (LPP), optimization of ratios of criteria gives more insight into the situation than optimizing (LPP).

Fractional programming has received remarkable attention in the literature. Ref. [5] gives a survey on fractional programming which covers applications as well as major theoretical and algorithmic developments. Thus, ratio functions arise in economic applications when efficiency measures of a system is to be optimized or in approaching a stochastic programming problems. Gilmore and Gomory [7] discussed a cutting stock problem in paper industry and showed that under the given circumstances, it is important to minimize the ratios of wasted and used amount of raw material instead of just minimizing the amount of wasted material. This leads to a linear fractional program.

Charne's and Cooper [3] showed that a linear fractional programming problems can be optimized by solving two linear programs, Zionts [14] had shown that if the linear fractional programming problem (LFPP) has a finite optimal solution, then whether a feasible solution to LFPP has a positive or negative denominator, it is sufficient to solve only one of the equivalent linear program depending on the sign of denominator. Craven [5] discussed in details LFP. There exists several methodologies [9, 10, 11, 12] to solve bilevel fractional programming problems and MLFPP. This paper presents a goal programming approach to solve MLFPP. Recently Malhotra and Arora [10] have presented a priority based goal programming approach to solve bilevel fractional programming. In this paper this approach is extended.

This paper is organized as follows. Section 2 defines the goal programming and discusses the case when the objective function is linear fractional. Section 3 gives the multi-level fractional programming and the method to convert this problem into a goal programming problem. Section 4 and 5 gives the algorithm and the flow chart. In order to facilitate the comprehension of the algorithm, a numerical example, is presented in Section 6. Finally a summary is given in Section 7.

2. Goal Programming

Goal programming (GP) is one of the approaches to solve the multiple objective programming problems.

In most of the multiple objective decision making problems, the objectives are competitive, incommensurable and often conflicting in nature. Such problems involve trade-off relations among the objectives to get the "optimal compromise solution". Goal programming introduced by Charnes and Cooper in 1961 [4] appeared as a robust tool to solve a linear multiobjective programming (MOP) problem given by

(P2) $\quad \text{Max } Z = CX$

$\quad\quad$ subject to $AX \le b$

$\quad\quad\quad\quad X \ge 0$

where $X \in R^n$, $Z = (z_1, z_2, \ldots., z_K)$ is the vector of objectives. C is a K x n matrix of objectives. A is an $m \times n$ matrix and $b \in R^m$. In GP the distance between the objective function vector Z and an aspiration level vector Z^* is minimized. The aspiration level is either determined by the DM or is taken as $Z^* = (z_1^*, z_2^*, \ldots., z_k^*)$ where z_k^* is the optimal value of z_k subject to the set of constraints in (P2)

General pre-emptive GP model to solve (P2) is given by

(P3) $\quad \text{Min } Z = \{ \sum_{k \varepsilon P_i} w_k g_k(d_k^-, d_k^+), i = 1,2,\ldots,I\}$

$\quad\quad$ subject to $AX \le b$

$$c_k X + d_k^- - d_k^+ = z_k^*$$

$$X \ge 0$$

$$d_k^-, d_k^+ \ge 0, \quad d_k^-.d_k^+ = 0, \quad \forall k = 1,2,\ldots,K$$

where d_k^-, d_k^+ are deviational variables and w_k are their weights, $g_k(d_k^-, d_k^+) = d_k^-$ in case of maximizing z_k, $g_k(d_k^-, d_k^+) = d_k^+$ in case of minimizing z_k and $g_k(d_k^-, d_k^+) = d_k^- + d_k^+$ when $z_k = z_k^*$ is required. c_k is the k^{th} row vector of matrix C. I is the number of priority levels and $k \varepsilon P_i$ means that k^{th} goal is in the i^{th} priority level.

Let the goal objective at the i^{th} priority level be a linear fraction given by

$$\frac{e_i X + \alpha_i}{f_i X + \beta_i}$$

where $e_i^T, f_i^T \in R^n, \alpha_i, \beta_i$ are scalars and $f_iX + \beta_i > 0$. Then the goal constraint corresponding to the i^{th} priority level is given by

$$\frac{e_i X + \alpha_i}{f_i X + \beta_i} + d_i^- - d_i^+ = z_i^*$$

i.e. $e_i X + \alpha_i + d_i^- (f_iX + \beta_i) - d_i^+ (f_iX + \beta_i) = z_i^* (f_iX + \beta_i)$

where z_i^* is the aspiration level for the objective goal, d_i^-, d_i^+ are the deviational variables. Then the fractional goal programming problem is

(P4) $\text{Minimize}\left[\sum_{k \in P^i} d_k^- + d_k^+, \quad k = 1, 2, \ldots K\right]$

subject to

$AX \le b$

$e_k X + \alpha_k + d_k^- (f_kX+\beta_k) - d_1^+ (f_kX+\beta_k) = z_k^* (f_kX+\beta_k)), k = 1,2,\ldots,K$

Let $d_k^-(f_k X + \beta_k) = D_k^-$

and $d_k^+(f_k X + \beta_k) = D_k^+$

(P4) is equivalent to the following goal programming problem

(P5) $\text{Minimize}\left[\sum_{k \in P^i} D_k^- + D_k^+, \ k = 1, 2, \ldots K\right]$

subject to $AX \le b$

$$(e_k X + \alpha_k) + D_k^- - D_k^+ = z_k^* (f_k X + \beta_k), \quad k = 1,2,\ldots,K$$

$$X,\ D_k^-,\ D_k^+ \ge 0, D_k^-.D_k^+ = 0, \quad \forall k = 1,\ldots.,K\ d_k^-$$

(P5) is a linear goal programming problem. The optimal values of d_k^-, d_k^+ in (P4) can be obtained from the optimal values of D_k^-, D_k^+ in (P5) based on the following results.

Let $S = \{X \in R^n \mid AX \le b, X \ge 0\}$ denote the feasible region.

Theorem 1 [8] : If the minimum of problem (P4) and (P5) occur at X_1 and X_2 respectively, where X_1 and X_2 are in S, then

(i) $d_i^- = 0$ if and only if $D_i^- = 0$.

(ii) $d_i^+ = 0$ if and only if $D_i^+ = 0$.

Theorem 2 [8] : If the minimum of problem (P4) and (P5) occur at X_1 and X_2 respectively where X_1 and X_2 are in S, then

(i) $\frac{D_i^-}{f_i X_1 + \beta_i} \le d_i^- \le \frac{D_i^-}{f_i X_2 + \beta_i}$, if $d_i^- > 0$

(ii) $\frac{D_i^+}{f_i X_1 + \beta_i} \le d_i^+ \le \frac{D_i^+}{f_i X_2 + \beta_i}$, if $d_i^+ > 0$

Thus we conclude that

(i) If $D_i^- = 0$ and $D_i^+ = 0$ in the optimal solution of the related problem (P5) then the optimal values of d_i^- and d_i^+ are also zero and the goal is completely achieved.

(ii) In case $D_i^- > 0$ in the optimal solution of the related GPP (P5) then $d_i^- > 0$ and the optimal value of d_i^- can be obtained by solving the following problem

Minimize $d_i^- = \text{Minimize} \frac{D_i^-}{f_i X + \beta_i}$

subject to

$$(e_i X + \alpha_i) - z_i^*(f_i X + \beta_i) = D_i^-$$

$$X \in S$$

Similar argument holds if $D_i^+ > 0$

3. GP Approach to MLFPP

Consider the multi-level programming problem with fractional objectives as follows:

(P6) $$\max_{X_1} f_1(X) = \frac{C_1 X + \alpha_1}{D_1 X + \beta_1}$$

$$\max_{X_2} f_2(X) = \frac{C_2 X + \alpha_2}{D_2 X + \beta_2}$$

$$\vdots$$

$$\max_{X_K} f_K(X) = \frac{C_K X + \alpha_K}{D_K X + \beta_K}$$

subject to $X \in S$

where S denotes the feasible region.

Solve the MLPP (P6) with the first level objective function f_1. Let $X^{11}, X^{12}, ..., X^{1K_1}$ be its optimal solutions.

Let $f_1(X^{11}) = f_1^{11}$. Then find the value of the second level DM objective function f_2 at these points and arrange them in descending order say

$$f_2(X^{11}) \geq f_2(X^{12}) \geq \ldots \geq f_2(X^{1K_1})$$

i.e. $$f_2^{11} \geq f_2^{12} \geq \ldots \geq f_2^{1K_1}$$

Introduce the aspiration levels $X_1^{11}, f_1^{11}, f_2^{11}$ for the variables X_1 and first and second level DMs objective function and solve the following GPP using pre-emptive goal programming technique.

(P7) Min $P_1\,(d_1^- + d_1^+)$

Min $P_2\,(d_2^- + d_2^+)$

Min $P_3\,(d_3^-)$

subject to

$$X_1 + d_1^- - d_1^+ = X_1^{11}$$

$$\frac{C_1X + \alpha_1}{D_1X + \beta_1} + d_2^- - d_2^+ = f_1^{11}$$

$$\frac{C_2X + \alpha_2}{D_2X + \beta_2} + d_3^- - d_3^+ = f_2^{11}$$

$$AX \leq b$$

$$X \geq 0$$

$$d_k^-, d_k^+ \geq 0 \qquad d_k^-.d_k^+ = 0, \qquad k = 1, 2, 3$$

where goals at priority P^1 and priority P^2 are taken to be absolute. Problem (P7) is equivalent to the following problem:

(P8) Min $P_1\,(d_1^- + d_1^+)$

Min $P_2\,(D_2^- + D_2^+)$

Min $P_3\,(D_3^-)$

subject to

$$X_1 + d_1^- - d_1^+ = X_1^{11}$$

$$C_1X + \alpha_1 + D_2^- - D_2^+ = f_1^{11}(D_1X + \beta_1)$$

$$C_2X + \alpha_2 + D_3^- - D_3^+ = f_2^{11}(D_2X + \beta_2)$$

$$AX \le b$$

$$X \ge 0$$

$$d_1^-,\ d_1^+ \ge 0 \qquad d_1^-.d_1^+ = 0,$$

$$D_k^-,\ D_k^+ \ge 0 \qquad D_k^-.D_k^+ = 0, \qquad k = 2,3$$

where $D_k^-, D_k^+,\ k = 2, 3$ are the deviational variables given by

$$D_2^- = d_2^-(D_1X + \beta_1)$$

$$D_2^+ = d_2^+(D_1X + \beta_1)$$

$$D_3^- = d_3^-(D_2X + \beta_2)$$

$$D_3^+ = d_3^+(D_2X + \beta_2)$$

We solve the problem (P8) using pre-emptive goal programming method.

The solution to the problem (P8) is either (i) feasible or (ii) infeasible, where feasible solution means that the goals at the priority level P_1 and P_2 are fully achieved i.e. the objective function values of goals at P_1 and P_2 are zero.

In case (i) $D_3^- = 0$ and $D_3^+ = 0$ and have X^{11} as the solution of the second level.

In case (ii) there are 3 possibilities

(a) If $D_3^- > 0$ then X^{11} is the solution of the second level.

(b) If $D_3^- = 0,\ D_3^+ = 0$ then also X^{11} is the solution of the second level.

(c) If $D_3^+ > 0$ then X^{11} is not the solution of the second level, repeat the process with the next alternate solution X^{12} taking $X_1 = X_1^{12}$, $f_1 = f_1^{12}$, $f_2 = f_2^{12}$. Continue this process till either some X^{ij}, j = 1, ..., K_1 turns out to be the solution to the second level or none of X^{ij}; j = 1, ...,K_1 is the solution of the second level.

In the later case, find the next best solution of the problem (P1). Let the next best value of f_1 be f_1^2 at the points X^{21}, X^{22}, ..., X^{2K_2}. Find the value of f_2 at these points and arrange them in descending order.

Let $f_2^{21} \geq f_2^{22} \geq \ldots \geq f_2^{2K_2}$. Repeat the process with the points X^{2j}, j = 1, 2, ..., K_2. Continue till some extreme point turns out to be the solution of the second level. Since the set of extreme points is finite, the process converges in a finite number of steps.

After finding the solution of the first two levels say X^{11}, consider the third level DMs objective function and find the value of the objective function f_3 at the solution of first two levels say f_3^{11} and consider the preemptive GP model of three level programming problem at X^{11} as

(P9) $\text{Min}\, P_1(d_1^- + d_1^+)$

$\text{Min}\, P_2(d_2^- + d_2^+)$

$\text{Min}\, P_3(d_3^- + d_3^+)$

$\text{Min}\, P_4(d_4^- + d_4^+)$

$\text{Min}\, P_5(d_5^-)$

subject to

$$X_1 + d_1^- - d_1^+ = X_1^{11}$$

$$\frac{C_1X + \alpha_1}{D_1X + \beta_1} + d_2^- - d_2^+ = f_1^{11}$$

$$X_2 + d_3^- - d_3^+ = X_2^{11}$$

$$\frac{C_2X + \alpha_2}{D_2X + \beta_2} + d_4^- - d_4^+ = f_2^{11}$$

$$\frac{C_3X + \alpha_3}{D_3X + \beta_3} + d_5^- - d_5^+ = f_3^{11}$$

$Ax \leq b$

$X \geq 0$

$d_k^-, d_k^+ \geq 0 \qquad d_k^-.d_k^+ = 0 \quad k = 1,2,3,4,5$

Problem (P9) is equivalent to the following problem:

(P10) $\text{Min}\, P_1(d_1^- + d_1^+)$

$\text{Min}\, P_2(D_2^- + D_2^+)$

$\text{Min}\, P_3(d_3^- + d_3^+)$

$\text{Min}\, P_4(D_4^- + D_4^+)$

$\text{Min}\, P_5(D_5^-)$

subject to

$X_1 + d_1^- - d_1^+ = X_1^{11}$

$C_1X + \alpha_1 + D_2^- - D_2^+ = f_1^{11}(D_1X + \beta_1)$

$X_2 + d_3^- - d_3^+ = X_2^{11}$

$C_2X + \alpha_2 + D_4^- - D_4^+ = f_2^{11}(D_2X + \beta_2)$

$C_3X + \alpha_3 + D_5^- - D_5^+ = f_3^{11}(D_3X + \beta_3)$

$AX \le b, \qquad X \ge 0$

$d_k^-, d_k^+ \ge 0 \qquad d_k^-.d_k^+ = 0 \qquad k = 1,3$

$D_k^-, D_k^+ \ge 0 \quad D_k^-.D_k^+ = 0, \quad k = 2,4,5$

where $D_k^-, D_k^+,\ k = 2,4,5$ are the deviational variables given by

$D_2^- = d_2^-(D_1X + \beta_1)$

$D_2^+ = d_2^+(D_1X + \beta_1)$

$D_4^- = d_4^-(D_2X + \beta_2)$

$D_4^+ = d_4^+(D_2X + \beta_2)$

$D_5^- = d_5^-(D_3X + \beta_3)$

$D_5^+ = d_5^+(D_3X + \beta_3)$

Solve the problem (P10) using pre-emptive goal programming method. We have three cases:

(a) If $D_5^- = 0$ and $D_5^+ = 0$, then X^{11} is the solution of the three level programming problem

(b) If $D_5^- > 0$ then X^{11} is the solution of the three level programming problem.

(c) If $D_5^+ > 0$ then X^{11} is not the solution of the three level programming problem and repeat the process.

After finding the solution of the first three levels, move to the next level and continue the process till all levels are included.

Here the solution process first starts by considering two levels and obtaining the solution of the first two levels and then we move to the next lower level and so on.

The solution procedure is summarized in the following goal programming algorithm:

4. Goal Programming Algorithm

Step 1 Let i = 1

Step 2 Solve the problem with the first level DMs objective function i.e. solve the problem

(P11) $$\max_{X_1} f_1(X) = \frac{C_1X + \alpha_1}{D_1X + \beta_1}$$

subject to $AX \le b,\ X \ge 0$

Let $X^{1j} = (X_1^{1j}, X_2^{1j}, \ldots, X_K^{1j})$, j = 1,...,$K_1$, be its optimal solution.

Let $f_1(X^{11}) = f_1^{11}$ and go to step3.

Step 3 Find the value of 2nd level DMs objective function f_2 at these points and arrange them in descending order say

$$f_2(X^{11}) \ge f_2(X^{12}) \ge \ldots \ge f_2(X^{1K_1})$$

$$\text{i.e. } f_2^{11} \ge f_2^{12} \ge \ldots \ge f_2^{1K_1}$$

Step 4 Set j = 1

Step 5 Convert the decision variables X_k, k = 1, ...,i and the objective functions of the first (i + 1)th DMs into goals and consider the following goal programming problem

(P12) **Min** $P_{2t-1}(d_{2t-1}^{-}+d_{2t-1}^{+})$ t = 1, ..., i

Min $P_{2t}(d_{2t}^{-}+d_{2t}^{+})$ t = 1, ..., i

Min $P_{2t+1}(d_{2t+1}^{-})$

subject to

$X_t + d_{2t-1}^{-} - d_{2t-1}^{+} = X_t^{ij}$ t = 1, 2, ..., i

$\frac{C_tX+\alpha_t}{D_tX+\beta_t} + d_{2t}^{-} - d_{2t}^{+} = f_t^{ij}$, t = 1, 2, ..., i

$\frac{C_{i+1}X+\alpha_{i+1}}{D_{i+1}X+\beta_{i+1}} + d_{2i+1} - d_{2i+1}^{+} = f_{i+1}^{ij}$

$Ax \le b$

$X \ge 0$

$d_t^{-}, d_t^{+} \ge 0$ $d_t^{-}.d_t^{+} = 0,$ $t = 1,2,3,...,2i+1$

Problem (P12) can be rewritten as

(P13) Min $P_{2t-1}(d_{2t-1}^{-}+d_{2t-1}^{+})$ t = 1, 2 ...., i

Min $P_{2t}(D_{2t}^{-}+D_{2t}^{+})$ t = 1, 2,...., i

Min $P_{2t+1}(D_{2t+1}^{-})$

subject to

$X_t + d_{2t-1}^{-} - d_{2t-1}^{+} = X_t^{ij}$ t = 1, 2, ..., i

$C_tX+\alpha_t + D_{2t}^{-} - D_{2t}^{+} = f_t^{ij}(D_tX+\beta_t)$, t = 1, 2, ..., i

$C_{i+1}X+\alpha_{i+1} + D_{2i+1}^{-} - D_{2i+1}^{+} = f_{i+1}^{ij}(D_{i+1}X+\beta_{i+1})$

$Ax \le b,$ $X \ge 0$

$d_{2t-1}^{-}, d_{2t-1}^{+} \ge 0$ $d_{2t-1}^{-}.d_{2t-1}^{+} = 0, t = 1,...,i$

$D_{2t}^{-}, D_{2t}^{+} \geq 0 \qquad D_{2t}^{-}.D_{2t}^{+} = 0, \; t = 1,\ldots, i$

$D_{2t+1}^{-}, D_{2t+1}^{+} \geq 0 \qquad D_{2t+1}^{-}.D_{2t+1}^{+} = 0,$

Step 6 Solve the problem (P13) by pre-emptive goal programming method. If X^{ij} is the feasible solution of GPP (P12) then go to step10, else go to step 7.

Step 7 Since the problem is infeasible there are two possibilities: (i) If $D_{2i}^{-} > 0$, $D_{2i+1}^{+} = 0$, $D_{2i+1}^{-} \geq 0$, then X^{ij} is the solution for the first $(i+1)^{th}$ levels and go to step 10.

(ii) If $D_{2i}^{-} > 0$ and $D_{2i+1}^{+} > 0$ then go to step 8.

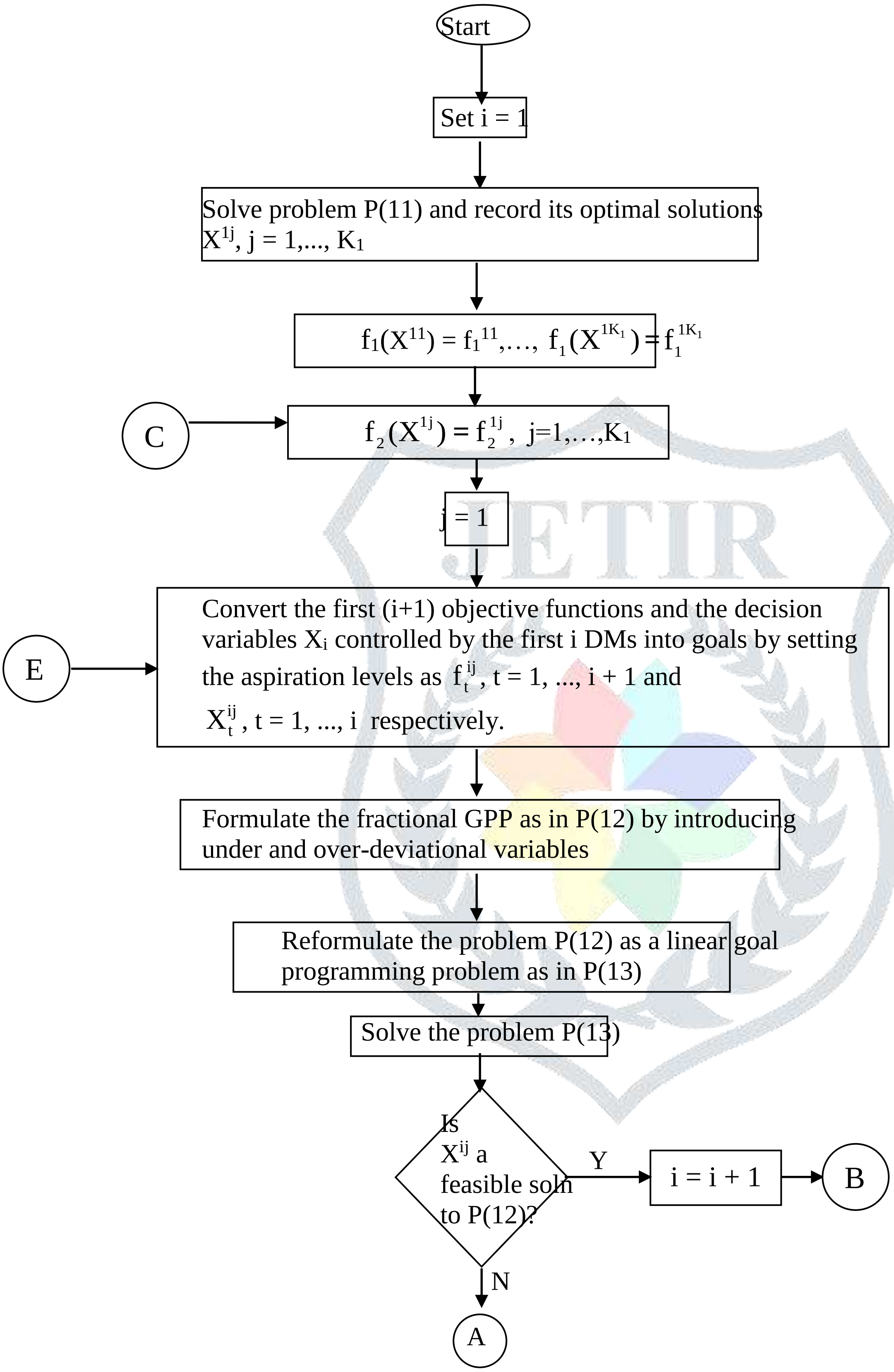
Step 8 If $j = K_i$, go to step 9 else set $j = j + 1$, $i = 1$ and go to step 5.

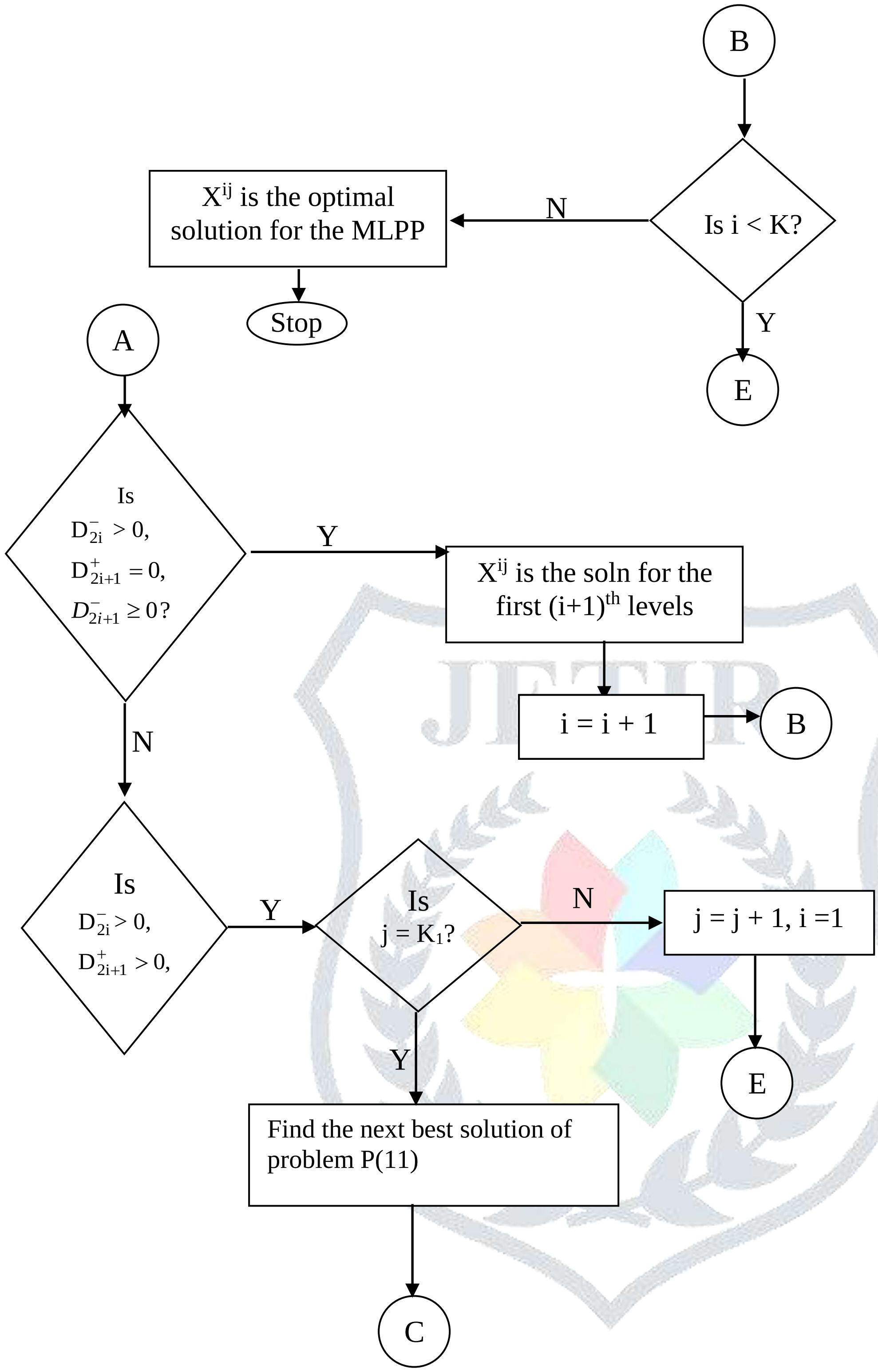
Step 9 Find the next best solutions of problem (P11) and repeat with this solution.

Step 10 Set $i = i + 1$, if $i < K$ then go to step 5 and if $i = K$, stop. X^{ij} is the optimal solution of the MLPP.

5. Flow Chart

The flowchart of the above algorithm is shown in fig.1.





6. Example

$$\underset{x_1}{\text{Max}}\ f_1(x) = \frac{x_1 + 2x_3}{x_1 + x_3 + 1}$$

$$\underset{x_2}{\text{Max}}\ f_2(x) = \frac{x_1 + 2x_2 + 4}{2x_1 + x_2 + x_3 + 1}$$

$$\underset{x_3}{\text{Max}}\ f_3(x) = \frac{x_1 + x_2 - x_3 + 1}{x_1 + x_2 + 3}$$

subject to

$$x_1 + 2x_2 + x_3 \le 6$$
$$2x_1 + x_2 - x_3 \le 4$$
$$2x_2 + 3x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

Solve the problem (P1) given by

(P14) Max $f_1(x) = \dfrac{x_1 + 2x_3}{x_1 + x_3 + 1}$

subject to.

$$x_1 + 2x_2 + x_3 \le 6$$
$$2x_1 + x_2 - x_3 \le 4$$
$$2x_2 + 3x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

Optimal solution of problem (P1) is (0, 0, 2) and the maximum value of f_1 is 4/3.

$f_2(0, 0, 2) = 4/3$

Therefore preemptive GP model of first two level programming problem with the point (0, 0, 2), $f_1 = 4/3$, $f_2 = 4/3$ is

(P15) Min $P_1\,(d_1^- + d_1^+)$

Min $P_2\,(d_2^- + d_2^+)$

Min $P_3\,(d_3^-)$

subject to

$$x_1 + d_1^- - d_1^+ = 0$$

$$\frac{x_1 + 2x_3}{x_1 + x_3 + 1} + d_2^- - d_2^+ = 4/3$$

$$\frac{x_1 + 2x_2 + 4}{2x_1 + x_2 + x_3 + 1}\ d_3^- - d_3^+ = 4/3$$

$$x_1 + 2x_2 + x_3 \le 6$$

$$2x_1 + x_2 - x_3 \le 4$$

$2x_2 + 3x_3 \le 6$

$x_1 , x_2 , x_3 \ge 0$

$d_i^- , d_i^+ \ge 0, \; d_i^- . d_i^+ = 0 \;\; \forall\, i = 1, 2, 3.$

The first two objectives are considered absolute.

Problem (P15) is equivalent to the following problem:

(P16) Min $P_1 \,(d_1^- + d_1^+)$

Min $P_2 \,(D_2^- + D_2^+)$

Min $P_3 \,(D_3^-)$

subject to

$x_1 + d_1^- - d_1^+ = 0$

$x_1 + 2x_3 + D_2^- - D_2^+ = 4/3(x_1 + x_3 + 1)$

$x_1 + 2x_2 + 4 + D_3^- - D_3^+ = 4/3(2x_1 + x_2 + x_3 + 1)$

$x_1 + 2x_2 + x_3 \le 6$

$2x_1 + x_2 - x_3 \le 4$

$2x_2 + 3x_3 \le 6$

$x_1 , x_2 , x_3 \ge 0$

$d_1^- , d_1^+ \ge 0 \quad d_1^- . d_1^+ = 0$

$D_i^- , D_i^+ \ge 0 \quad D_i^- . D_i^+ = 0 \;\; \forall\, i = 2, 3.$

where $D_2^- = d_2^- (x_1 + x_3 + 1)$ $\qquad D_2^+ = d_2^+ (x_1 + x_3 + 1)$

$D_3^- = d_3^- (2x_1 + x_2 + x_3 + 1) \quad D_3^+ = d_3^+ (2x_1 + x_2 + x_3 + 1)$

Solve (P16) using pre-emptive GP technique.

The solution (0, 0, 2) is infeasible as $D_3^+ \ge 0$.

Since the alternate solution to (P14) does not exist, so consider the next best solution (3, 0, 2) and solve (P16) with constraint (i) replaced by $x_1 + d_1^- - d_1^+ = 3$ and the constraint (ii) and (iii) respectively replaced by

$$x_1 + 2x_3 + D_2^- - D_2^+ = 7/6(x_1 + x_3 + 1)$$

$$x_1 + 2x_2 + 4 + D_3^- - D_3^+ = 7/9(2x_1 + x_2 + x_3 + 1)$$

The solution to this problem is infeasible.

Since the alternate solution does not exist, so move to the next best solution (4/3, 2, 2/3) of the first level. Formulate and solve the GPP for the first two levels. On solving we find that (4/3, 2, 2/3) is a feasible solution for the two levels.

Consider the third level. Formulate and solve the GPP.

On solving we find that (4/3, 2, 2/3) is not the feasible solution for the three level program.

Move to the next best solution (2/3, 8/3, 0) of the first level and formulate the pre-emptive GPP

(P17) Min $P_1 (d_1^- + d_1^+)$

Min $P_2 (D_2^- + D_2^+)$

Min $P_3 (D_3^-)$

subject to.

$$x_1 + d_1^- - d_1^+ = 2/3$$

$$x_1 + 2x_3 + D_2^- - D_2^+ = 2/5(x_1 + x_3 + 1)$$

$$x_1 + 2x_2 + 4 + D_3^- - D_3^+ = 2(2x_1 + x_2 + x_3 + 1)$$

$$x_1 + 2x_2 + x_3 \leq 6$$

$$2x_1 + x_2 - x_3 \leq 4$$

$$2x_2 + 3x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

$$d_1^-, d_1^+ \geq 0 \quad d_1^- . d_1^+ = 0$$

D_i^- , $D_i^+ \geq 0$ $D_i^- . D_i^+ = 0$ $\forall$ i = 2, 3.

where $D_2^- = d_2^- (x_1 + x_3 + 1)$ $D_2^+ = d_2^+ (x_1 + x_3 + 1)$

$D_3^- = d_3^- (2x_1 + x_2 + x_3 + 1)$ $D_3^+ = d_3^+ (2x_1 + x_2 + x_3 + 1)$

The solution to this problem is feasible.

Move to the third level.

$f_3(2/3, 8/3, 0) = 13/19$

Solve the GPP

(P18) Min $P_1 (d_1^- + d_1^+)$

Min $P_2 (d_2^- + d_2^+)$

Min $P_3 (d_3^- + d_3^+)$

Min $P_4 (d_4^- + d_4^+)$

Min $P_5 (d_5^-)$

subject to

$x_1 + d_1^- - d_1^+ = 2/3$

$$\frac{x_1 + 2x_3}{x_1 + x_3 + 1} + d_2^- - d_2^+ = 2/5$$

$x_2 + d_3^- - d_3^+ = 8/3$

$$\frac{x_1 + 2x_2 + 4}{2x_1 + x_2 + x_3 + 1} + d_4^- - d_4^+ = 2$$

$$\frac{x_1 + 2x_2 + 4}{2x_1 + x_2 + x_3 + 1} + d_5^- - d_5^+ = 13/19$$

$x_1 + 2x_2 + x_3 \leq 6$

$2x_1 + x_2 - x_3 \leq 4$

$2x_2 + 3x_3 \leq 6$

$x_1 , x_2 , x_3 \geq 0$

d_i^- , $d_i^+ \geq 0$, $d_i^- . d_i^+ = 0$ $\forall$ i = 1, 2, 3,4,5.

Problem (P18) is equivalent to the following problem:

(P19) Min P_1 ($d_1^- + d_1^+$)

Min P_2 ($D_2^- + D_2^+$)

Min P_3 ($d_3^- + d_3^+$)

Min P_4 ($D_4^- + D_4^+$)

Min P_5 (D_5^-)

subject to

$x_1 + d_1^- - d_1^+ = 2/3$

$x_1 + 2x_3 + D_2^- - D_2^+ = 2/5(x_1 + x_3 + 1)$

$x_2 + d_3^- - d_3^+ = 8/3$

$x_1 + 2x_2 + 4 + D_4^- - D_4^+ = 2(2x_1 + x_2 + x_3 + 1)$

$x_1 + x_2 – x_3 + 1 + D_5^- - D_5^+ = 13/19(x_1 + x_2 + 3)$

$x_1 + 2x_2 + x_3 \leq 6$

$2x_1 + x_2 - x_3 \leq 4$

$2x_2 + 3x_3 \leq 6$

$x_1 , x_2 , x_3 \geq 0$

d_i^- , $d_i^+ \geq 0$, $d_i^- . d_i^+ = 0$ $\forall$ i = 1, 3.

D_i^- , $D_i^+ \geq 0$ $D_i^- . D_i^+ = 0$ $\forall$ i = 2, 3.

where $D_2^- = d_2^- (x_1 + x_3 + 1)$ $D_2^+ = d_2^+ (x_1 + x_3 + 1)$

$D_4^- = d_4^- (2x_1 + x_2 + x_3 + 1)$ $D_4^+ = d_4^+ (2x_1 + x_2 + x_3 + 1)$

$D_5^- = d_5^- (x_1 + x_2 + 3)$ $D_5^+ = d_5^+ (x_1 + x_2 + 3)$

Solving we get the feasible solution as (2/3, 8/3, 0).

Thus (2/3, 8/3, 0) is the optimal solution of the given problem with optimum values of f_1, f_2, f_3 as 2/5, 2, 13/19.

7. Summary and concluding remarks:

This paper has proposed a multi-level fractional decision making problem with linear constraints and a goal programming method for solving this problem.

The proposed goal programming method gives an efficient solution for MLFPP keeping the hierarchy intact. The higher level DMs provides the preferred values of the decision variables under their control and the target value of their objective functions to the next level DM to formulate a goal programming problem equivalent to the given multi-level programming problem. The solution process first starts by considering the first two levels and obtaining the solution of the first two levels and then moving to the next lower level till all the levels are included. An illustrated numerical example has been provided to demonstrate the proposed solution method.

8. References

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