EXCLUDED FUZZY SET

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Abstract:

In this paper fuzzy singleton has been used instead of Wong's fuzzy points. The reason for this is that fuzzy singletons reduce to ordinary ones and the foremost reason is that every fuzzy generalization should be formulated in such a way that it contains the corresponding ordinary set-theoretic notion as a special case. The introduction of the sierpinski space revealed an important department from ordinary set theory. In other words it reveals the connection between a fuzzy set and its complement in terms of fuzzy singletons.

Introduction:

Zadeh's introduction of the concept of fuzzy set in the realm of mathematics has inspired many mathematicians to generalize the main concepts and structures of present day mathematics into the framework of fuzzy setting. Chang[1] formulated the concept of fuzzy topological space and its basic notions. In order to construct counter-examples in ordinary topology, one has to make use of somewhat pathological spaces such as the sierpinski space. It seemed quite reasonable that similar fuzzy topological spaces will be needed in the further development of fuzzy topology.

The key object in the construction of fuzzy topological spaces such as the sierpinski space, the included (excluded) fuzzy singleton topology and the included (excluded) fuzzy set topology, is the fuzzy singletons as proposed by Palaniappan[2].

Let X be a ordinary nonempty set called the universe of discourse. A fuzzy set A on X is a mapping on X into the closed interval [0, 1] associating with each element X of X its grade of membership $A(x)$ in A.

Fuzzy Singletlon :

A fuzzy set on X is a fuzzy singleton if it takes the value 0 for all points X in X except one. The point in which a fuzzy singleton takes the non-zero value will be called the support of the singleton and the corresponding element of]0, 1[is its value. Fuzzy singletons will be denoted by lower case letters p, p, ……… .

Excluded Fuzzy Set Topology :

Let X be a non-empty set and A be a fuzzy set on X. Let τ_A be the subclass of $P(X)$ given by

$$\tau_A = \{0 : 0 \in P(X) \wedge (0 = X \vee A \subseteq C \circ 0)\}$$

Then τ_A is a topology on X and is usually called as the excluded fuzzy set topology on X.

In this section our aim is to study fuzzy sierpinski space for the class of Lowen's fuzzy topological space and will study about its appropriateness.

In fuzzy topology the analogue of the Sierpinski topological space was suggested first by Kerre, which we have studied in the first section of this chapter. E.E.Keree and E.G.Manes suggested the fuzzy Sierpinski space only for the class of fuzzy topological spaces in the sense of C.L.Chang.

In this section, we observe that a reasonable fuzzy Sierpinski space can be obtained for the class of fuzzy topological spaces in the sense of Lowen by modification of the one suggested by Kerre. Interestingly this fuzzy Sierpinski space turns out to be non-topological. There is one often asked question. Are there natural fuzzy topological spaces which are non-topological? It gives the answer in affirmative.

Most of the fuzzy topological concepts used here, are standard and are proposed by Chang and Lowen.

If X is a set and $I = [0, 1]$, then $\tau \subseteq I^X$ is called a fuzzy topology in the sense of Chang. If $0, 1 \in \tau$ and τ is closed under finite infima and arbitrary suprema whereas τ is called as fuzzy topology in the sense of Lowen, if in addition to the above, τ also contains all constats. If (X, τ) is a fuzzy topological space (fts) in Lowen's sense, it is called topological if there is a topology τ on X such thet $\tau = \emptyset(\tau)$, where

$\emptyset(\tau) = \{f : (X, \tau) \rightarrow I : f \text{ is 1.s.c}\}$

If $p \in I^X$ is non-zero exactly at one point of X, p is called a fuzzy **singleton** of X. If $A \in I^X$ and $p \leq A$, we write $p \in A$.

1. Let $X = \{0, 1\}$ and $p \in I^X$ be a fuzzy singleton of X. Then, the family

   $\tau_p = \{A \in I^X : p \in A\} \cup \{\emptyset\}$

can very easily be visualized as a fuzzy topology in the sense of Chang's. Then (X, τ_p) be called a **fuzzy Sierpinski space** in the sense of Kerre's.

Here we recall that if $(2, S)$ is the standard Sierpinski topological space then it has the following properties:

(i) (2, S) 'classifies' open subsets i.e. a subset A of a topological space (X, τ) is open iff $\chi_B : (X, \tau) \rightarrow (2, S)$ is continuous

(ii) $(2, S)$ is connected

(iii) $(2, S)$ is a **'Sierpinski object'** in the category TOP of topological spaces in the sense of Manes (1976) i.e. for any topological space (X, τ), τ is the induced topology on X with respect to all continuous function $\tau : (X, \tau) \rightarrow (2, S)$.

(iv) TOP can be characterized in a suitable class of categories via the existence of Sierpinski object of Mane's (page 155)

It seems desirable that 'right' fuzzy Sierpinski space must possess fuzzy topological analogs of the above four properties. Obviously Kerre's fuzzy Sierpinski spaces do not meet these requirements (regarding the analog of property (ii)). We will use fuzzy connectedness as defined by Hutton (1980). That is a fuzzy topological space (fts) (X, τ) in the sense of Chang's is said to be connected if X has no closed opent (clopen) fuzzy set except 0 and 1. The fuzzy Sierpinksi space as proposed by A.K. Srivastave in (1989) repairs this situation and defined it as that fuzzy topological space (I, u) where $u = \{0, 1, 1\}$. Here $I : I \rightarrow I$ is identity function. Moreover, if we consider the more general L-fuzzy topological spaces say of Hutton, where L is a suitable lattice then for such L-fts, an obvious modifications of "(I, u) acts as L-fuzzy Sierpinski space (just by replacing I by L)

Here we observe that (I, u) is a fuzzy topological space in the sense of Chang's only. So, it is natural to inquire is there is a suitable fuzzy Sierpinski space for the class of Lowen's fuzzy topological space.

Here we see that this is indeed possible. Let V be the fuzzy topology on I, in the Lowen's sense generated by $\{0, I\} \cup \{C : I \to 1\}$: C is a constant function}. Call $(I, V]$ the fuzzy Sierpinski space. The first theorem, which We wish to study makes use of fuzzy connectedness. Evidently, Hutton s concept of fuzzy connectedness will not work for no fuzzy topological space (X, τ) in Lowen,s sense can be fuzzy connected in Hutton's sense because of constant in τ. We therefore call an fuzzy topological space, fuzzy connected, if it has no non-constant closed-open (clopen) set. Henceforth all fuzzy topological space will be in the sense of Lowen.

Theorem :

If (I, V) is fuzzy connected, injective and $A \in I^X$ is fuzzy open in a fuzzy topological space (X, τ) if and only if $A : (X, \tau) \to (I, V)$ is fuzzy continuous.

Proof :

Let (X, τ) is a fuzzy topological space in the sense of Lowen. Let $I = [0, 1]$ and $A \in I^X$ is fuzzy open in an fts (X, τ). Further we suppose that (I, V) is fuzzy connected and injective.

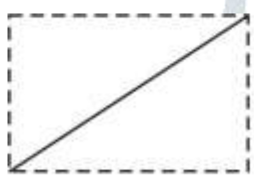
Fig.1


Fig.2

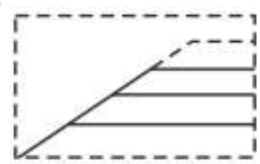
Fig.3

Let us assume that the sub basic open fuzzy sets of (I, V) are of the types shown in figure (i) and (ii) and the basic open fuzzy sets of (I, V) are of type shown in figure (ii) and (iii).

Then, it is clear that (I, V) has no non-constant clopen fuzzy set. And so (I, V) must be fuzzy connected.

Next, if $X \subseteq Y$ is any subspace of an fts (Y, S) and $f : (X, \tau) \to (I, V)$ be any fuzzy continuous function (τ being the subspace fuzzy topology), then $f = f^{-1}(I) \in \tau$. Since X is a subspace, there must exist some $g \in S$ such that g : $X = f$. Thus f has an extension g to Y and so (I, V) must be injective. Finally, let $A \in I^X$ and (X, T) be a fuzzy topological space. If A is fuzzy continuous, $A^{-1}(I) = A$ must be open.

Conversely, if A is open then for any sub basic open fuzzy set $B \in V$ then $A^{-1}(B)$ is either A or a constant and hence in each case, an open fuzzy set Thus A must be fuzzy continuous.

In order to study the next two results, we require some categorical concepts whose definition can be found in Manes (1976).

These concepts are 'categories of sets with structures', an 'optimal family of morphisms', an 'optimal lift of a family of maps', and 'Sierpinski object' in a category of sets with structures.

The category FTOP of fts (fuzzy topological spaces) and fuzzy continuous functions is a category of sets with structures.

Optimal Fuzzy Continuous Maps :

A family $f_i : (X, \tau) \to (X_i, \tau_i)$ of fuzzy continuous maps is said to be optimal if an only if τ is the fuzzy topology induced on X by the family f_i.

Optimal Lift :

An optimal lift of a family $f_i : X \to (X_i, \tau_i)$ always exists and is the fuzzy topology on X induced by $f_i.' s$.

Theorem :

To show that (I, V) is a Sierpinski object in FTOP.

Proof :

Let (X, τ) be a fuzzy topological space and let $F = \{f : (X, \tau) \to (I, V): f$ is fuzzy continuous$\}$. We shall show that τ is the smallest fuzzy topology on X making all members of F fuzzy continuous. Let τ' be another such fuzzy topology on X. If $A \in \tau$ then $A : (X, \tau) \to (I, V)$ is fuzzy continuous and so $A \in F$. But then $A : (X, \tau') \to (I, V)$ must be fuzzy continuous and so $A^{-1} (1) = A \in \tau'$. Thus $\tau \subseteq \tau'$. Hence the result.

Sierpinski Object :

Let $\subseteq$ is a category of sets with structures. Then, we say that $\subseteq$ has a Sierpinski object (I, V), call $A \in I^X$ (fuzzy) open in $(X, s) \in \subseteq$ if and only if $A : (X, s) \to (I, V)$ is admissible.

Subbase :

A family $A_i \in I^X$ of fuzzy open sets has a subbase if and only if $\{A_i (X, s) \to (I, V)\}$ is optimal.

Theorem :

A category $\subseteq$ of sets with structures is isomorphic to FTOP iff there exists a "fuzzy sierpinski space" (I, V) in $\subseteq$ with underlying set [0, 1] satisfying the following five conditions

(i) Every family $f_i : X \to (I, V)$ has an optimal lift.

(ii) The map sup : $(I, V)^X \to (I, V)$ is admissible in $\subseteq$ for every set X where $(I, V)^X = (I^X, \tau)$ where τ is the optimal lift of all projections $I^X \to (I, V)$.

(iii) The map in $f : (I, V)^X \to (I, V)$ is admissible in $\subseteq$ for every finite set X.

(iv) (I, V) is a sierpinski object in $\subseteq$.

(v) If S is a subbase and A is fuzzy open then A is the union of finite intersections of elements of S.

Proof:

First of all we shall show the FTOP satisfies the five conditions i.e. (i) to (v). In view of the observations made above, however only three conditions (i), (iii) and (v) need verifications. Obviously by the statement that a family $A_i \in I^X$ of fuzzy open sets of fts (X, s) be called a basis if and only if $\{A_i : (X, s) \to (I, V)\}$ is an optimal family we merely mean that s should be the smallest fuzzy topology on X containing these A_i', s. However, this is well know, So FTOP satisfies the condition (v). Next, we recall that the product fuzzy topology on I^X has a natural subbase, that is $\{p_{x-1}(\mathrm{U}) : \mathrm{U} \in V : x \in X\}$. Also $p_{x-1}(\mathrm{U}) = p_x$ for U = I and $p_{x-1}(\mathrm{U}) = \underline{\tau}$ for $\mathrm{U} = \underline{\tau}$, where $\underline{\tau} \in I^X$ is the τ-valued constant.

Evidently, $\mathrm{Sup}^{-1}(I) = I$ o Sup = Sup. Also, for . $f \in I^X, \mathrm{Sup}\, f = \underset{X \in X}{\mathrm{Sup}}\, f(x) = \underset{X \in X}{\mathrm{Sup}}\, p_x(f)$

This shows that Sup = $V\{p_X : x \in X\}$. Since p_x is fuzzy open, Sup is fuzzy open. Also, $\mathrm{Sup}^{-1}(\tau) = \tau$. This shows that Sup is fuzzy continuous.

Similarly, we can show that inf (I) = inf = $\wedge\{p_x : x \in X\}$. Hence $\inf^{-1}(I)$ is fuzzy open if X is finite. Thus, we can say that inf : $(I, V)^X \to (I, V)$ is admissible if X is finite FTOP, therefore, satisfies (ii) and (iii) also.

Now the proof can be completed by producing suitable bijection between $C(X)^*$ and $F(X)$, were $F(X)$ is the family of all fuzzy topologies on X. To each $s \in C(X)$, we associate $\tau_S \in F(X)$ where τ_S is given by

$\tau_S = \{A \in I^X : A : (X, s) \to (I, V) \text{ is } \subseteq \text{ admissible}\}$.

We must check that τ_S is indeed a fuzzy topology. Let $(A_j : j \in J\} \subseteq \tau$. Let us consider the map $f : (X, s) \to (I, V)^J$ defined by $f\,(X)(j) = A_j(X)$, $x \in X$, $j \in J$. Then f is ⊆ admissible as (I, v) is sierpinski object and $A_i = p_i$ o f, (where $p_i : (\mathrm{I}^X, \tau) \to (I, \mathrm{v})$ is the i^{th} projection) is ⊆ admissible. Now Sup of = ∪ $(A_j : \mathrm{j} \in J\}$ .Hence ∪$(A_J : j \in J\}$ is fuzzy open, Thus τ_S is a fuzzy topology. Now, finally we should show that if $f : (X, s) \to (Y, w)$ is C-admissible then $f : (\mathrm{X}, \tau_S) \to (Y, \tau_W)$ is fuzzy continuous. Finally, we assume that $B \in \tau_W$ then $B : (Y, w) \to (I, \mathrm{v})$ is $\underline{C}$ admissible and therefore B o $f = f^{-1}(B)$ and therefore, it follows that $f^{-1}(B) \in \tau_S$.

Theorem :

Show that there do not, exists topology, T on I, such that v = $w(T)$. That (I, V) is non-topological fuzzy topological space.

Proof :

If possible let us assume that, there be a topology T on I such that v = $w(T)$. Then $I \in w(T)$ and so $I^{-1}[\tau, 1] = (\tau, 1] \in T \ \forall\ \tau \in [0, 1]$.

Now let us consider the function $f : (I, T) \to I$ defined by $f(x) = x^2$.

Then $f^{-1}[\tau, 1] = \{x \in I : x^2 \in [\tau, 1]\} = \left\{x \in I : x \in \left(\sqrt{\tau}, 1\right)\right\} = \left(\sqrt{\tau}, 1\right)$

Hence $f^{-1}[\tau,1]\in T \;\forall\; \tau\in[0,1]$ showing that *f* is 1. s.c. In other words, $f \in w(\tau)$. But from figure (i), (ii) and (iii) in the proof of the theorem or otherwise, it is clear that $f \notin$ v. This contradicts the supposition v = $w(\tau)$.

Hence the result.

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