Optimality conditions and Algorithm For Quadratic Programs

Dr. Anuradha Sharma (Associate Professor)
University of Delhi INDIA
anuradhagaurs@yahoo.co.in

Abstract:. This article presents a new approach for the development of conditions under which optimal solution is obtained for a special class of non-linear programs, specifically Quadratic programs. The methodology is quite user friendly and involves lesser number of iterations and computational work. An algorithmic development and demonstration of the algorithm is shown with an illustrative example.

Keywords: Quadratic Programming; Indefinite ;Quadratic Programming, Non-Linear Programs, Optimality Conditions

1. Introduction

Escalated Interest in non-linear programming grew with the growth of linear programming. Many researchers developed necessary and sufficient conditions for the existence of an optimal solution to a non-linear programming problems. A subclass of non-linear programming problem is a one in which the objective function is non-linear but the constraints are all linear giving rise to a variety of problems depending up on the nature of the objective function.

When the objective function is given by

$$Z = \dot{\alpha} + CX + X^{T}QX ,$$

problem is called quadratic program. Various algorithm for solving mathematical approach systematically to local optimal solution. Under certain assumptions, a local optima can be shown to be global optima. When the constraint set is a convex polyhedron and the objective function is convex/concave, local minimum/maximum is also a global minimum. Single objective decision making method reflect an earlier and simpler era. The world has become more and more complex as one enters the information age. We can find that almost every important real world problem involves more than one objective.

A non-basic variable x_i change either its value current breakpoint value either in the left direction or in the right direction.

$X^* = (X_B{*} , X_N{*})$.

$c_B{*}$; $d_B{*}$ are the corresponding Z_{1j} and Z_{2j}

Moving to another improved solution

Let $X^0_{B^*}$ be the current basic feasible solution corresponding to basis B^*. If corresponding to this basis $\xi_i^- \le 0$ and $\xi_i^+ \ge 0$, $i = 1, ..., p$ then X^0_B is optimal to (IQP).

Suppose there exists at least one non-basic variable x_r, corresponding to which optimality condition is not satisfied, $\xi_r^- > 0$ or $\xi_r^+ < 0$. Hence, the variable x_r will be the entering variable.

Case (a): Let $\varepsilon_r^+ < 0$ and x_r be the corresponding entering variable. In this case, the non-basic variable x_r undergoes a change ϕ_r^* .Let the new solution so obtained be $(\hat{X}_r) = (\hat{x}_r)$

The value of ϕ_r^* should be restricted so that

$\varepsilon^{\mu(B^*)}_B \le x_{B^*} \le \varepsilon$ and $x_r \le (\varepsilon^{\lambda_r+1}$.

The new solution $(\hat{X}_r)^*$ is a feasible extreme point, provided $\varphi_r^* = min\{\eta_1, \eta_2, \eta_3\}$

$^{+1} - \varepsilon^{\lambda_r}\}$

Case (b): Let $\varphi_r = \eta_1$ that is, $\varphi_r = y_{s^*r}$ for some $s_r \in I$

. This implies that $x_{B_r^*}$ $\mu(B_{s^*}+1)$

becomes basic and $x_{B_{s^*}}$ departs from the basis and attains the value at break point εB_{s^*} .

The change in the values of the objective functions $Z_{i1}(\hat{X})^*$, $Z_{i2}(\hat{X})^*$ and the basic variables $r$component of $C_B{*}$

If $\xi_r^+ \ge 0$, then ϕ_r^* is the maximum possible change in the value of the objective functions, given by equation (4). If $\xi_r^+ < 0$, then there is a possibility of increasing the value of ϕ_r^* by redefining $\mu_{B_{s^*}}$ as $\mu_{B_{s^*}}+1$. This generates new value of ϕ_r^* . The process is repeated until optimality condition is attained.

Case (i): Let $\phi_r^* = \eta_2$

$\varphi^*_r =$

$\mu(B*)$

$_{B}s^* - B s^*$

for some s_r

$\in I^*$.

This implies that $x_{B r^*}$ becomes basic and departs from basis. The change in the values of objective functions and variables are given by equations (3) and (5).If $\xi^+ \geq 0$, then optimal solution If $\xi^+ < 0$, then change the value redefining μ_{Bs^*} as $\mu_{Bs^*} - 1$ in η_2. A new value of ϕ^*_r is generated and the process is repeated.

1. Algorithm for solving piecewise quadratic program

Step 1: Consider piecewise indefinite quadratic programming problem (IQP).

Step 2: Find initial basic feasible solution X^0_{*B} of (IQP).

Step 3: Check the optimality conditions for X^0_{*B} If $\xi^-_i \leq 0 \ \forall i = 1, 2, ...p$, then X^0_{*B} is optimal solution. Otherwise, go to step 4 or step 5 according as $\xi^+_i < 0$

Step 4: Let $\xi^+ < 0$. Choose a non-basic variable x_r as the entering variable.

Go to next step.

Step 5: Let $\xi^-_r > 0$. Then, a non-basic variable x_r is chosen as the entering variable.

Go to step 6.

Step 6: Update new basis and examine optimality condition for new basic feasible condition. If new solution is optimal, it is optimal solution else follow step 4 .

2. Illustrative example

Let x_1, x_2, and x_3 be the time (in months) which an investor can consider to be with company A1, B1 and C1

$$12x_1 + 3x_2 + x_3 = 155$$

$$5x_1 + x_2 + x_3 = 110$$

objective is find an investor should invest money in three companies to maximize profit.

$$Z_{11}(x_1) = \begin{cases} 3x_1 + 0, & 0 \le x_1 \le 2; \\ 5x_1 - 4, & 2 \le x_1 \le 6 \end{cases}$$

$$Z_{12}(x_1) = \begin{cases} -x_1 + 5, & 0 \le x_1 \le 2; \\ 3x_1 - 3, & 2 \le x_1 \le 6 \end{cases}$$

$$Z_{21}(x_2) = \begin{cases} 2x_2 + 5, & 0 \le x_2 \le 2; \\ 6x_2 - 3, & 2 \le x_2 \le 4 \end{cases}$$

$$Z_{22}(x_2) = \begin{cases} -x_2 + 3, & 0 \le x_2 \le 2; \\ x_2 - 1, & 2 \le x_2 \le 4 \end{cases}$$

$x_3 + 4, \quad 0 \le x_3 \le 2$

$$Z_{31}(x_3) = 2x + 2,$$

$$2 \leq x \leq 3$$

$$3x_3 - 1,$$

$$-\quad \leq$$

$$3 \leq x_3 \leq 6$$

$$Z_{32}(x_3) = 2x_3 + 5,$$

$$0 - \quad x_3 \leq 2 \leq$$

$$x + 3,$$

$$2 \; x \leq 3$$

$$x_3 - 3,$$

$$3 \leq x_3 \leq 6$$

Solution. The indefinite quadratic programming is defined as

$$Max \; \sum_{i=1}^{3} Z_i(x_i)$$

subject to

$$2x_1 + 3x_2 + x_3 = 15;$$

$$x_1 + x_2 + x_3 = 10;$$

$$0 \leq x_1 \leq 6,$$

$$0 \leq x_2 \leq 4,$$

$$0 \leq x_3 \leq 6$$

Introducing artificial variables x_4 and x_5, then the problem reformulated above can be written as

$$Max \; \sum_{i=1}^{5} Z_i(x_i)$$

subject to

$$2x_1 + 3x_2 + x_3 + x_4 = 15;$$
$$x_1 + x_2 + x_3 + x_5 = 10;$$

$$0 \leq x_1 \leq 6,$$
$$0 \leq x_2 \leq 4,$$
$$0 \leq x_3 \leq 6,$$
$$x_4 \geq 0,$$
$$x_5 \geq 0$$

where $Z_4(x_4) = (-M.x_4 + 0)(0.x_4 + 0)$

Select x_1 as an entering variable. For departing variable, calculate $\varphi^*_r = Min(\eta_1, \eta_2, \eta_3)$, $\varphi^*_r = \min\left\{\frac{15}{2}, \frac{10}{1}, 2\right\}$ Now, $\varphi^*_r = 2$, corresponding to η_3, therefore, x_1 will increase 0 to 2, but will continue as non basic variable. x_4=11,$x_5 = 8$. select x_1 as an entering variable.

Calculate $\varphi^*_r = \min\left\{11, 8, 4\right\}$ The solution is given by (6,4,0,1)optimality condition is satisfied.

Hence, solution for quadratic program is (5, 0, 5, 0, 0) with
$maxZ = 640$.

Here x_1 =25, x_2 =14, x_3 = 5. This means that person should invest his money with company A and C for 5 months and with company B, he should not invest at all. The profit from investing money in three companies is $Rs.$640 million. The above example is solved using LINGO 17.0. The optimal solution is (25,14,5)

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