SOME CHARACTERIZATIONS BASED ON DISCRETE ANALOGUE OF FISHER INFORMATION MEASURE

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Abstract
In this paper characterizations based on minimal amount of information are discussed. The result of Johnstone and MacGibbon (1984) related to the characterization of Poisson distribution has been generalized by Kapoor (1987) by replacing random variable x by $\alpha(x)$. Important characterizations of discrete probability distributions like Binomial, Poisson and Negative-binomial distributions are obtained. Other non-standard distributions can also be characterized by using this approach.

Index Terms – Characterization, Fisher Information Measure, Cramer-Rao Inequality

Contribution of the work

The paper will review in detail the result of Johnstone & MacGibbon. Characterization of the binomial and negative-binomial distributions has already been arrived at by Kapoor (1987). This result has been extended using the same approach. The function of random variable X has been given particular value and characterization of the geometric distribution derived at.

Notations

In the paper, P is the class of 'probability distribution', E is expectation, $Li(\theta)$ is likelihood function, $\frac{\partial \log Li(\theta)}{\partial\theta}$ is the first derivative of log likelihood function, Var (T) is the variance of unbiased estimator T.

Introduction

Fisher Information measures the amount of information of unknown parameter θ as available from the random variable X of a distribution. Fisher Information gives information of unknown parameters based on sample observation and hence it is a criterion for 'reckoning parameters'. Fisher Information measure for discrete analogue is defined as the following:
For each $p \varepsilon P$ (the class of distributions concentrated on the set of non-negative integers).

$$I(P) = \sum_{x:p_x>0} \frac{(p_x - p_{x-1})^2}{p_x};\ p_{-1} = 0$$

The concept of Fisher Information measure is derived from 'Cramer-Rao Inequality' which gives a lower bound to the variance of unbiased estimator, thereby helps in the selection of MVU estimators as can be seen from the following inequality

$$V(T) \geq \frac{1}{E\left[\left(\frac{\partial \log Li(\theta)}{\partial\theta}\right)^2\right]} \quad \text{or Var}(T) \geq \frac{1}{I(\theta)}$$

i.e. Var (T) is the inverse of Fisher Information measure.
Frechet (1943), Darmois (1945), Aitken and Silverstone (1942) also arrived at 'Cramer-Rao bound' independently.

Characterization and Fisher Information
Kagan, Linnik, and Rao (1973) characterized some continuous distributions based on shift and scale parameters. Since the focus of the paper is on discrete distributions hence they will be discussed in detail.

Johnstone and MacGibbon (1984) defined the discrete analogue of Fisher's Information with reference to the set of non-negative integers. Fisher's Information measure is independent of θ and if origin is shifted from x to $x-\theta$ then also Fisher Information measure will remain same if $\theta = 0$. Johnstone & Mac Gibbon also used Fisher Information at $\theta = 0$. Using the minimality of Fisher Information measure, Kapoor (1987) generalized their result and characterized the birth and death process.

Result of Johnstone and MacGibbon (1984) is as follows:

If $x \in X_0, E(x) < \infty$ with $\{x: p_x > 0\}$
then $I_X \geq (\text{Var}(x))^{-1}$
for $I_X = (\text{Var}(x))^{-1}$ can be attained in case when x is a Poisson random variable.
They characterized the Poisson distribution based on minimizing the discrete analogue of the Fisher Information measure.

Kapoor (1987) generalized the above result by replacing x by any function of x, say $\alpha(x)$. Then he characterized the birth and death process. Giving particular values to $\alpha(x)$ characterization of distributions were obtained.

To characterize the birth and death process Kapoor (1987) proved the following theorem to generalize the result arrived at by Johnstone & MacGibbon (1984).

Theorem
If $x \in X_0 = \{x: p_x > 0\}$ with left extremity equal to zero and $\alpha(x)$ be any non-negative function on $\Delta_0 = (0,1,2,...)$ such that $\alpha(0) = 0, \alpha(x) > 0$ and $\lambda = E(\alpha(x)) < \infty$,
if $E\alpha(x^2)$ is finite
then $I_X \geq \psi^2(\text{Var}(\alpha(x)))^{-1}$ ... (1)
where ψ is finite and is defined by $\psi^2 = \sum_{\{x:p_x>0\}}(p_x - p_{x-1})(\alpha(x) - \lambda)$
Equality in (1) is achieved
iff $p_x = (\prod_{r=1}^{x} \frac{\lambda}{\alpha(r)})p_0$ where $p_0, p_1 \ldots p_n > 0$, for $n > 0$ and $p_{n+1} = 0$
This is a 'probability distribution function' of birth and death process (Kapoor 1987:20)

Proof as given by Kapoor (1987:21-22)

$$\psi^2 = \sum_{\{x:p_x>0\}}(p_x - p_{x-1})(\alpha(x) - \lambda)$$

Implying that $I_X \geq \psi^2(\text{Var}(\alpha(x)))^{-1}$ [Using 'Cauchy Schwartz Inequality']

Applying the argument of Johnstone & MacGibbon (1984), Kapoor (1987) arrived at the probability distribution function of birth and death process as follows:

$$p_x = \left(\prod_{r=1}^{x} \frac{\lambda}{\alpha(x)}\right) p_0$$

This is characterization of the birth and death process.

Characterization of Geometric distribution

$$p_x = \prod_{r=1}^{x} \frac{\lambda}{\alpha(r)} p_0 \qquad \ldots(2)$$

Particular value was given to $\alpha(r)$ in equation (2)

Consider $\alpha(r) = \dfrac{\lambda}{(1-\theta)\theta^{\frac{2r}{x(x+1)}}}$

$$p(x) = \prod_{r=1}^{x} (1-\theta)\theta^{\frac{2r}{x(x+1)}} p_0$$

$$= (1-\theta)^x \theta^{\sum_{r=1}^{x} \frac{2r}{x(x+1)}} p_0$$

$$= (1-\theta)^x \theta^{\frac{2}{x(x+1)} \sum_{r=1}^{x} r} p_0$$

$$= (1-\theta)^x \theta p_0$$

This is a geometric distribution.

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