

A New Approach For Quadratic/Linear Bilevel Programs Using Differential Evolution

Dr. Anuradha Sharma Associate Professor, University of Delhi, India.

Abstract: This paper presents a quadratic/Linear Bilevel Programming Problem (QLBPP) having single decision maker at both levels i.e. upper level (first level) and lower level(second level).The objective function of first level being quadratic and lower level objective being a linear function over a convex polyhedral set as the constraint set.The solution approach follows a hierarchical strategy i.e. the objective function at upper level is optimized first and then for a given set of variables under the control of first level, the objective function at second level is optimized. In the decision making situation, decision deadlock occurs very often. This algorithm deals by changing the randomly generated initial population into an initial population satisfying the constraints to improve the efficiency of the algorithm to deal with the constraint which utilizes N number of D-dimensional parameter vectors as a population for each generation G. The main advantage of the proposed algorithm is that it avoids the introduction of penalty function in order to deal with the given constraint set. The algorithm presented in this paper aims at global optimization and lesser computations to solve decentralized systems . The algorithm presented in this article incorporates a parallel direct search method which utilizes N number of D-dimensional parameter vectors as a population for each generation G. The algorithm proposed in this paper bears good convergence properties and controls only few variables which are fixed. The(QLBPP) under consideration is at first reduced to a single level program using KKT conditions and then decomposed into two separate problems which is plausible because the decomposed problems don't carry common variables thereby highlighting the advantages of the proposed method. Despite of the fact that feasible region of (QLBPP) is a convex polyhedron(irrespective of linear/nonlinear nature of objective functions),the BLPs are neither continuous everywhere nor convex because the objective function value of the first level problem is implicitly determined by the follower's objective function which is neither linear nor differentiable. The proposed approach is demonstrated numerically.

Keywords: Bilevel Programming; Quadratic programming; Differential Evolution; Separable Problems; Hierarchical optimization; NP-Hard.

Mathematics subject classification 1991
primary: 90C20; secondary: 90C26

1.INTRODUCTION

A general optimization problem is to select n decision variables x_1, x_2, \dots, x_n from a feasible region in such a way that it optimizes(minimize/maximize) the objective function $f(x_1, x_2, \dots, x_n)$ of decision variables.The problem is called a non-linear programming problem(NLP) if the objective function is non-linear and/or the feasible region is determined by non-linear constraints. Interest in non-linear programming has grown simultaneously with the growth of linear programming. Kuhn and Tucker developed a necessary and sufficient condition for the existence of an optimal solution to a non-linear programming problem, which is basis for further development in the field.A simple subclass of nonlinear programming problem is a one in which the objective function is non-linear but the constraints are all linear.This gives rise to a variety of problems depending up on the nature of the objective function. When the objective function is given by

$$Q(x,y) = (p_1x + q_1y) + \frac{1}{2} (x, y)^T \begin{bmatrix} p_{11} & q_{11} \\ p_{12} & q_{12} \end{bmatrix} (x, y)$$

the problem is called quadratic programming problem. For given set of linear constraints, optimal solution of a general nonlinear programming problem may not always exist at an extreme point. In fact, a study of the nature of the objective function is necessary to predict this. In this paper a Quadratic/Linear Bilevel Programming problem (QLBPP) is considered having single decision maker at both levels i.e. upper level (first level) and lower level (second level). The objective function of first level being quadratic and lower level objective being linear function with linear constraints. The solution approach follows a hierarchical strategy i.e. the upper level objective function is optimized first and then for a given set of variables under the control of first level, the objective function at a given set of variables under the control of first level, the objective function at second level is optimized. In the decision making situation, decision deadlock occurs very often. The execution of decisions is sequential, starting from top and moving to lower level. The decision maker at each level tries to maximize its own benefits, but is affected by decisions of decision makers at other levels through externalities. The upper level decision maker first sets his goals and then asks the subordinate level about its solution which is computed in isolation. Bilevel Programming Problems are characterized by planner at a certain hierarchical level determining his own objective and constraint space by successive level partially under cooperation. The first level decision maker called the Leader optimizes its objective function first and for a given choice of variables under its control. Further, each of the decision makers optimize their objectives. The order imposed on the choice of the decision reflects the hierarchical nature of the problem. The Leader masters the information of follower's objective and constraints, while follower optimizes its objective after Leader's strategy is declared. In other words, if the lower level decision maker is not satisfied by the decision of upper level decision maker, we move to find the next best solution to first level problem. The bilevel programming problem is NP-hard. It is neither convex nor continuous in nature.

Algorithms for solving a general mathematical program approach systematically to a local optimal solution. Under appropriate assumptions local optima can be shown to be a global optima. For instance when the constraint set is a convex polyhedron and the objective function is convex (concave), the local minima (maxima) is also a global minima (maxima). Non-linear Programming Problems are NP-Hard [6,7,12].

Many real life problems applications of BLP problems can be observed viz. Candler and Townsley [4], Bard [2]. Stackelberg strategy was proposed by Abo Sinna [1] for bilevel decentralized system as well as multilevel programming Problems (MLP). BLPPs have been referred in [2,5,8,14,15,16,17] in non-linear cases, while [10,11,18] don't discuss non-linear BLPPs in detail. Solution to linear fractional bilevel programming problem using goal programming has been presented by Malhotra and Arora [11]. Here a quadratic/linear bilevel programming problem (QLBPP) is studied and an algorithm is proposed to solve it based on differential evolution (DE). This algorithm deals by changing randomly generated initial population into initial population satisfying the constraints to improve the efficiency of the algorithm to deal with constraints. The proposed algorithm takes into account less computational work and is easy to implement. Finally, the algorithm is supported numerically.

The paper is organized as follows. Section 1 gives a brief introduction of the problem under consideration. Section 2 presents related works in field of bilevel programming. Section 3 presents the mathematical formulation of (QLBPP). In section 4, the non-linear bilevel programming problem (QLBPP) is transformed into a single level problem with complementarity constraints by use of Kuhn-Tucker conditions for second level problem. In section 5, we present an algorithm to solve the (QLBPP) for the maximization type objective function by decomposing the given problem into two problems which is possible because of the presence of independent variables in the constraints. In Section 6 the computational performance of the procedure is evaluated. Finally, the concluding remarks are made in section 7.

2. APPLICATIONS AND RELATED WORKS

Let us consider a real life example of Bilevel Programming Problem in which the government is at the first level. During the planning period, the government sets certain goals. To achieve each of these goals, government formulates certain policy measures such as taxes and subsidies. Industries at the second level design their course of action keeping such policy measures under consideration in order to achieve their objectives. Many researchers tackled the problem by presenting both theoretical results and application. The algorithmic approaches developed so far can be classified into four categories: meta heuristics approach; Kuhn-Tucker conditions approach; fuzzy approach; vertex enumeration approach.

Pal and Moitra[13] formulated quadratic programming model to optimize problem by using Hamming distance[19]. Ignizio[9] applied linear approximation technique for non-linear goal programming model in the decision making process. He placed the decision variable at first priority level and objective goals at second level of priority irrespective of system constraints. However, these constraints play a vital role in such systems, it was observed that following this approach, undesirable solutions were obtained. So, this approach was not much appreciated due to several stages and transformation of variables for solving a BPP.

This paper presents an algorithm based on differential evolution(DE). The algorithm proposed in this paper bears good convergence properties and controls only a few variables which are fixed. This algorithm deals by changing the randomly generated initial population into an initial population satisfying constraints to improve the efficiency of the algorithm to deal with the constraints. The (QLBPP) under consideration is decomposed into two separate problems which is plausible because the decomposed problems don't carry common variables. Despite of the fact that the feasible region of (QLBPP) is a convex polyhedron, the BLPs are neither continuous everywhere (irrespective of linear/non-linear nature of objective functions) nor convex because the objective function value of the first level problem is implicitly determined by the follower's objective function which is neither linear nor differentiable. Mostly the traditional solution methodologies depend on the search space and are insufficient to tackle real life problems. Till date the algorithms developed so far include genetic algorithms[3,8,12], particle swarm optimization[10], simulated annealing algorithms [13].

3. MATHEMATICAL FORMULATION OF (QLBPP)

The (QLBPP) can be mathematically formulated as :

$$(QLBPP) \max_{x,y} Q(x,y) = (p_1x + q_1y) + \frac{1}{2} (x, y)^T \begin{bmatrix} p_{11} & q_{11} \\ p_{12} & q_{12} \end{bmatrix} (x, y)$$

where y solves

$$\begin{aligned} \max_y f_2(x,y) &= p_2x + q_2y \quad \text{for a given } x \\ \text{subject to } &A_1x + A_2y \leq b \\ &x, y \geq 0 \end{aligned}$$

where the first function is quadratic and second level function is linear in nature, x is an n_1 - dimensional column vector (the variables under control of first level) and y is an n_2 -dimensional column vector (the variables under the control of second level); p_1 and p_2 are n_1 -dimensional row vectors, q_1 and q_2 are n_2 -dimensional row vectors; A_1 is an $m \times n_1$ matrix and A_2 is an $m \times n_2$ matrix and b is an m -dimensional column vector. We assume that the polyhedron S defined by the common constraints is nonempty and bounded

4. Reduction of QLBPP Into a single level program

For reducing QLBPP into a single level program, Kuhn-Tucker conditions for the second level are employed.

The QLBPP is reduced to the following single level program

$$\begin{aligned}
 \text{(NLP1)} \quad \max_{x,y} Q(x,y) &= (p_1x + q_1y) + \frac{1}{2} (x, y)^T \begin{bmatrix} p_{11} & q_{11} \\ p_{12} & q_{12} \end{bmatrix} (x, y) \\
 \text{s.t.} \quad A_1x + A_2y + r &= b \\
 t A_2 - s &= q_2 \\
 r t = 0, s y &= 0 \\
 x, y, r, s, t &\geq 0
 \end{aligned}$$

Given the vector $v_{i,k}: i \in \{1,2,\dots,N\}$, we assume that the constraint space has m constraints and $y \in \mathbb{R}^{n_2}$. The vectors are encoded as $(m+n_2)$ real numbers

$$\begin{aligned}
 \text{i.e. } v_{i,k} &= (y_{i,k}, a_{i,k}) \\
 &= (y_{i1,k}, y_{i2,k}, \dots, y_{in_2,k}, a_{i1,k}, a_{i2,k}, \dots, a_{im,k})
 \end{aligned}$$

where $y_{i,k} = (y_{i1,k}, y_{i2,k}, \dots, y_{in_2,k})$, $a_{i,k} = (a_{i1,k}, a_{i2,k}, \dots, a_{im,k}) \geq 0$ being the vectors y and a in (NLP1) resp.

Hence, for a given $y_{i,k}$, the following cases need to be considered

Case(a) If $y_{ij,k} > 0$, then $s_j = 0$ (the j th variable of vectors in (NLP1)).

Case(b) If $r_{il,k} > 0$, then $t_l = 0$.

The above strategy is applied to each $v_{i,k}, i \in \{1,2,\dots,N\}$ and the problem (NLP1) reduces to

$$\begin{aligned}
 \text{(QLP2)} \quad \max_x p_1x \\
 \text{subject to} \\
 \cdot \quad A_1x = b - A_2 y_{i,k} - r_{i,k} \\
 t' A_2' - s' = q_2 \\
 x, t', s' \geq 0
 \end{aligned}$$

where t' & s' are non-zero variables of t & s resp.; A_2' being the rows of A_2 associated to t' .

The problem (QLP2) can be decomposed into the following two problems:

$$\begin{aligned}
 \text{(QLP3)} \quad t' A_2' - s' &= q_2 \\
 t', s' &\geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 \text{(QLP4)} \quad \max_x p_1x \\
 \text{subject to} \\
 \cdot \quad A_1x = b - A_2 y_{i,k} - r_{i,k} \\
 x \geq 0
 \end{aligned}$$

Such decomposition can be done since problems (QLP3) and (QLP4) don't have any common variables. Firstly, it is observed that in case (QLP3) is infeasible, then the target vector cannot be accessed, otherwise (QLP4) is solved which when infeasible, then $v_{i,k}$ cannot be accessed.

Let $(x_{i,k}, y_{i,k})$ be the optimal solution of (QLP3) corresponding to $v_{i,k}$.

5. ALGORITHMIC DEVELOPMENT

The algorithm presented in this paper aims at global optimization and lesser computations to solve decentralized systems. The algorithm presented in this article incorporates a parallel direct search method which utilizes N number of D -dimensional parameter vectors $V_{i,G} = (V_{i1,G}, V_{i2,G}, V_{i3,G}, \dots, V_{iD,G})$; $i=1,2,\dots,N$ (2) as a population for each generation G . More specifically, the basic approach in the algorithm is summarized in the following steps:

Step 1: *Initialization* : Consider a set of N accessible vectors $\{v_{i,0}: i=1,2,\dots,N\}$. In order to generate these vectors, the following problem

$$\begin{aligned} & \max_{x,y} \quad e x + q_2 y \\ & \text{s.to.} \quad A_1 x + A_2 y \leq b \\ & \quad \quad x, y \geq 0 \end{aligned}$$

where $e \in R^{n1}$ is a random row vector, is solved. The optimal solution being dependent on e , but $q_2 y$ has to satisfy optimality conditions. These solutions are extreme point solutions of the constraint space. When the solutions $(x_{i,0}, y_{i,0}), i=1,2,\dots,N$ are generated, they are transformed into N vectors using the transformation

$$a_{i,0} = b - (A_1 x_{i,0} + A_2 y_{i,0}), \text{ then } v_{i,0} = (y_{i,0}, a_{i,0}).$$

Further, the value of the objective functions of the first level is used for the fitness value of each vector, that is, $fitness(v_{i,0}) = Q(x_{i,0}, y_{i,0})$ and $fitness(v_{i,G}) = Q(x_{i,G}, y_{i,G})$.

The individuals generated in this way all satisfy the constraints of the problem (1).

Step2: In accordance to equation

$$w_{i,G+1} = v_{e1,G} + F \cdot (v_{e2,G} - v_{e3,G})$$

With random indices $e1, e2, e3 \in \{1,2,\dots,N\}$, integer mutually different and $F > 0$ (the weighting factor), the vectors $w_{i,G+1}, i=1,2,\dots,N$ are generated.

Step3: In case problem (QLP3) and (QLP4) are feasible for any $i: 1 \leq i \leq N$ and at the same time

$$fitness(z_{i,G+1}) > fitness(v_{i,G}), \text{ then set}$$

$$v_{i,G+1} = z_{i,G+1} \text{ else } v_{i,G+1} = v_{i,G}$$

Step4: In case the condition is satisfied or iteration number exceeds the maximal iteration, stop. Keeping the current best vector in the earlier performed iterations, the best solution so generated is the solution for (QLBPP).

6. NUMERICAL ILLUSTRATION

A numerical example is presented in order to demonstrate the capability so as to tackle variety of complexities and conditions occurring in problems. By making use of different random numbers, on each test instance, the algorithm was run 20 times.

$$\begin{aligned} \text{(QLBPP)} \quad & \max_{x,y} f_1(x,y) = 6x + 3y - x^2 - y^2 \\ & \text{where } y \text{ solves} \\ & \max_y f_2(x,y) = 4x + 5y \text{ for a given } x \\ & \text{subject to} \quad 2x + 3y \leq 12 \\ & \quad \quad x + y \leq 5 \\ & \quad \quad 2x + y \leq 6 \\ & \quad \quad 3x + 2y \leq 9 \\ & \quad \quad x, y \geq 0 \end{aligned}$$

The best solution of this problem occurs at $x=0.6$ and $y=3.6$ with $f_1= 1.08$ and $f_2 = 20.40$ for a population of size 20, crossover constant 0.8, weighting factor 0.9 over 100 generations by using the differential evolution algorithm proposed in this article, while over the same number of generations the optimal solution occurs at $x=0.62$ and $y=3.661$ with $f_1= 1.0775$ & $f_2 = 20.32$ by genetic algorithm method crossover weight 0.9 and mutation rate 0.1.

7. CONCLUSIONS

This article presents a non linear Quadratic/Linear Bilevel Programming Problem (QLBPP) which is solved using the concept of differential evolution. This algorithm deals by changing the randomly generated initial population into an initial population satisfying the constraints to improve the efficiency of the algorithm to deal with the constraints. The algorithm presented in

this paper aims at global optimization and lesser computations to solve decentralized systems. The main advantage of the proposed algorithm is that it avoids the introduction of penalty function in order to deal with the given constraint set. The algorithm presented in this article incorporates a parallel direct search method which utilizes N number of D-dimensional parameter vectors as a population for each generation G. The (QLBPP) under consideration is decomposed into two separate problem which is plausible because the decomposed problems don't carry common variables. The algorithm proposed in this paper makes use of differential evolution concept bearing good convergence properties and controls only a few variables which are fixed. The (QLBPP) under consideration is first transformed into single level program using KKT conditions and the resultant single level problem is decomposed into two separate problems which is plausible because the decomposed problems don't carry common variables. The algorithm presented in this paper aims at global optimization & lesser computations to solve decentralized systems. The algorithm presented here incorporates parallel direct search method. The algorithm presented in this paper is easy to implement and takes lesser computational efforts for solving nonlinear bilevel programs and give better results than the proposed genetic algorithms over the same number of generations and population size. The author hopes for its being helpful in the future study of non-linear bilevel programs. The algorithm is supported with a numerical example and a comparison is made with the genetic algorithmic method of solution to establish its better performance and its usefulness.

Acknowledgements: The author is thankful to the editor and referees for their valuable suggestions that helped in the presentation, execution and improvement of the article.

8. REFERENCES

- [1] Abo Sinna M.A., A bilevel non-linear multi-objective decision making under fuzziness, Journal of the Operational Research Society of India (OPSEARCH), 2001, 38(5): 484-495.
- [2] Bard J F , Falk J E ., An explicit solution to the multi-level programming problem. Computers & Operations Research, 1982, 9(1):77-100.
- [3] Calvete H I , Gale C., P.M. Mateo P.M., A new approach for solving linear bilevel problems using genetic algorithms. European Journal of Operational Research, 2008, 188:14-28.
- [4] Candler W., Townsley.R., A linear two-level programming problem. Computers & Operations Research, 1982, 9:59-76.
- [5] Chen Y. , Florian M., The Non-Linear Bilevel Programming Problem: formulations, regularity and optimality conditions, Optimization, 1995, 32(3), 193-209.
- [6] Dempe S.. Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints. Optimization, 2003, 52:333-359.
- [7] Edmunds T. and Bard J., Algorithms for non-linear bilevel mathematical problems, IEEE Transactions Systems, Man, and Cybernetics, 1991, 21(1): 83-89.
- [8] Hejazi S.R., Memariani A., Jahanshahloo G., Sepehri M.M., Linear bilevel programming solution by genetic algorithm. Computers & Operations Research, 2002, 29:1913-1925.
- [9] Ignizio J.P. Goal Programming and Extensions, (1976) Lexington, MA Lexington DC.
- [10] Kuo R.J., Huang C.C., Application of particle swarm optimization algorithm for solving bi-level linear programming problem. Computers & Mathematics with Applications, 2009, 58(4):678-685.
- [11] H.H. Darji, B. Shah & M.K. Jaiswal. "CONCEPTS OF DISTRIBUTED AND PARALLEL DATABASE", International Journal of Computer Science and Information Technology & Security (IJCSITS), ISSN:2249-9555 vol. 2, no. 6, Dec. 2012, No;224, pg. no: 1150 available at https://www.academia.edu/42308519/CONCEPTS_OF_DISTRIBUTED_AND_PARALLEL_DATABASE
- [12] Malhotra N. & Arora S., An algorithm to solve linear fractional bilevel programming problem via goal programming, Journal of Operational Research Society of India (OPSEARCH), 2000, 37(1): 1-13.

- [13] Osman M.S., Abd El-Wahed W.F., A solution methodology of bi-level linear programming based on genetic algorithm. *Journal of Mathematics and Statistics*, 2009, 5(4):352–359.
- [14] Pal B.B. and Moitra B.N., A fuzzy goal programming procedure for solving Quadratic bilevel programming problems, *International journal of intelligence Systems* 2003, 18(5) :529- 540.
- [15] Pan K.J., Chen H., Tan Y.J., Multi-exponential inversion of T_2 spectrum in NMR based on differential evolution algorithm, *Acta Physica Sinica (in Chinese)*, 2008, 57(9): 5956–5961.
- [16] Sahin K.H., Cirit A.R., A dual temperature simulated annealing approach for solving bilevel programming problems. *Computers and Chemical Engineering*, 1998, 23:11–25.
- [17] Savard G. & Gauvin J., The steepest descent direction for the non-linear bilevel programming problem, *Operations Research Letters*, 1994, 15(5): 265-272.
- [18] Storn R., Price K., Differential evolution: a simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization*, 1997, 11:341–359.
- [19] Wen U.P., Hsu.S.T., Algorithms for solving the mixed integer two level linear programming problem. *Computers & Operations Research*, (1990) 17(2):133-142.
- [20] Zeleny, M., Multiple criteria decision making, Mc Graw Hill, New York. (1982)

