# BETA- LENGTH BIASED PARETO DISTRIBUTION AND ITS PROPERTIES

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## ABSTRACT

Length-biased distributions are a special case of the more general form known as weighted distributions. The concept of weighted distribution is frequently used in studies related to reliability, survival analysis, analysis of family data, biomedicine, ecology, etc. In practice, weighted distributions are used and applied when observations from a sample are recorded with unequal probabilities. In this paper a Beta-Length Biased Pareto distribution has been proposed. Also, various mathematical properties of the BLBP distribution have been presented.

**Keywords:** Pareto Distribution, Size-biased distribution, Length Biased Distribution, Beta-Length Biased Pareto Distribution, , arcsine distribution, logbeta distribution

## 1.Introduction

Size-biased distributions are a special case of the more general form known as weighted distributions. The concept of weighted distribution is frequently used in studies related to reliability, survival analysis, analysis of family data, biomedicine, ecology, etc. In practice, weighted distributions are used and applied when observations from a sample are recorded with unequal probabilities. Statistical models that take into account such restrictions are known as weighted models. Rao (1965) presented a unified concept of weighted distribution and identified various sampling situations that can be modelled by weighted distributions. When observations are selected with probability proportional to their length, the resulting distribution is called length-biased. Length-biased distributions have been particularly found to be very useful in biometry, wildlife studies, and for the analysis of lifetime data. The statistical interpretation of length-biased distributions was originally identified by Cox (1969) in the context of renewal theory. Patil and Rao (1978) examined some general models leading to weighted distributions and showed how the weight w(x) = x occurs in a natural way in many sampling problems.

The Pareto distribution was first proposed by Wilfred Pareto (1906). This distribution finds wide ranged applications in economics, life testing studies, survival analysis, business and industrial product management, etc. A random variable X is said to follow Pareto distribution if its probability density function (PDF) is given by:

$$f(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}} \; ; x \geq \beta, \alpha, \beta > 0 \qquad (1.1)$$

Where β is the scale parameter and α is the shape parameter of the distribution. The mean of this distribution is given by,

$$E(X) = \frac{\alpha\beta}{\alpha - 1}, \alpha > 1$$

The probability density function (PDF) of Length Biased Pareto (LBP) distribution is given by,

$$h(x) = \frac{xf(x)}{E(x)}$$

i.e.,

$$h(x) = (\frac{\alpha-1}{\beta})(\frac{\beta}{x})^{\alpha} \quad (1.2)$$

The distribution function of LBP distribution is given by,

$$H(x) = P(X < x) = \int_{\beta}^{x} (\frac{\alpha-1}{\beta})\beta^{\alpha} t^{-\alpha} dt \quad (1.3)$$

i.e., $H(x) = 1 - (\frac{x}{\beta})^{-(\alpha-1)}$

## 2. Beta- Pareto Distribution

Eugene et al. (2002) and Jones (2004) proposed beta-generated distributions. Since then, a great variety of beta-generated distributions have been studied by several researchers. The beta-generated distribution applies the technique of constructing a univariate distribution function by incorporating in the beta distribution, another parent distribution function. The method essentially involves beginning with an initial baseline probability density function (PDF) f(x) and the corresponding cumulative distribution function (CDF) F(x), the beta generated family of distribution has the PDF given by,

$$g_F(x,m,n) = \frac{1}{B(m,n)} f(x)F(x)^{m-1}[1 - F(x)]^{n-1} \quad (2.1)$$

and the CDF of the beta-generated distribution is given by,

$$G_F(x,m,n) = \frac{1}{B(m,n)} \int_0^{F(x)} t^{m-1}(1-t)^{n-1} dt \quad (2.2)$$

Recently, Akinsete et al. (2008) proposed the Beta-Pareto distribution. Nanuwong and Bodhisuwan (2014) have studied the properties of the Length-Biased Beta-Pareto distribution. In this paper a new distribution namely the Beta- Length Biased Pareto distribution is proposed and some of its structural properties have been studied.

## 3. Beta- Length Biased Pareto Distribution

Here, the Length-Biased Pareto distribution is taken as the baseline distribution function with the PDF and the CDF given by equations (1.2) and (1.3) respectively. The CDF of the new distribution namely Beta-Length Biased Pareto distribution is given by,

$$G(x) = \frac{1}{B(m,n)} \int_0^{H(x)} t^{m-1}(1-t)^{n-1} dt \quad (3.1)$$

The corresponding PDF for the Beta-Length Biased Pareto (BLBP) distribution is given by,

$$g(x) = \frac{1}{B(m,n)} H(x)^{m-1}[1 - H(x)]^{n-1}H'(x), m,n > 0 \quad (3.2)$$

From the above equation (3.2), the PDF of the BLBP distribution is obtained as,

$$g(x) = \frac{\alpha-1}{\beta B(m,n)}\, [1-(\tfrac{x}{\beta})^{-(\alpha-1)}]^{m-1}\, (\tfrac{x}{\beta})^{-\alpha n+n-1} \quad (3.3)$$

It is easily verified that, $\int_{\beta}^{\infty} g(x)\,dx = 1$ and therefore $g(x)$ is the PDF of the beta generated BLBP distribution. Therefore, if a random variable X has the density function as in equation (3.3), we shall write X ~ BLBP $(\alpha,\beta,m,n)$.

The CDF of the BLBP distribution; $G(x)$ is given by the equation (3.1). It may be re-expressed as $G^*(x) = 1 - G(x)$. Then, we have,

$$G^*(x) = \int_{x}^{\infty} \frac{\alpha-1}{\beta B(m,n)}\, [1-(\tfrac{t}{\beta})^{-(\alpha-1)}]^{m-1}\, (\tfrac{t}{\beta})^{-\alpha n+n-1}\, dt, t \geq \beta \quad (3.4)$$

Let, $z = (\frac{t}{\beta})^{-(\alpha-1)}$, then the equation (3.3) becomes,

$$G^*(x) = \frac{1}{B(m,n)}\int_0^z (1-z)^{m-1} z^{n-1} dz$$

That is, we have

$$G^*(x) = \frac{B(z;n,m)}{B(m,n)}, \; 0 < z < 1 \quad (3.5)$$

Where, $B(z;n,m)$ is an incomplete beta function with $z = (\frac{t}{\beta})^{-(\alpha-1)}$.

We therefore have,

$$G(x) = 1 - \frac{B(z;n,m)}{B(m,n)}$$

Then by using the infinite series expansion for the incomplete beta function we get,

$$G(x) = 1 - \frac{z^n}{B(m,n)}\{\frac{1}{n} + \frac{1-m}{n+1}\, z + \frac{(1-m)(2-m)}{2!(n+2)}\, z^2 + \cdots \ldots \ldots \ldots \ldots \ldots ..\} \quad (3.6)$$

When $m = 1, n = 1$, the above equation (3.6) gives the CDF of the Length Biased Pareto random variable.

## 4. Some Structural Properties of BLBP Distribution

In this section some important structural properties of the BLBP distribution are discussed. These are:

**Case 1:** Taking $m = 1, n = 1$, in the PDF of the BLBP distribution as given by equation (3.3), we obtain,

$$g(x) = (\frac{\alpha-1}{\beta})(\frac{\beta}{x})^{\alpha} \quad (4.1)$$

Which is the PDF of the Length Biased Pareto distribution.

**Case 2:** Taking $m = 1$ in the PDF of the BLBP distribution as given by equation (3.3), we obtain,

$g(x) = \frac{c\beta^c}{x^{c+1}}$, where $c = n(\alpha - 1)$ (4.2)

Therefore, it follows that when $m = 1$, the BLBP distribution reduces to Pareto distribution with parameters $c = n(\alpha - 1)$ and $\beta$.

**Case 3:** If a random variable X ~ BLBP$(\alpha, \beta, m, n)$ then the random variable Y=$-(\frac{X}{\beta})^{-(\alpha-1)}$, has the arcsine distribution when $m = \frac{1}{2}, n = \frac{1}{2}$, and $\alpha = 4n - 1, n \in N$.

Using the transformation method, it is easy to show that the random variable Y has the arcsine density function given by,

$f(y) = \frac{1}{\pi\sqrt{y(y-1)}}$ where $0 < y < 1$ (4.3)

**Case 4:** If a random variable X ~ BLBP$(\alpha, \beta, m, n)$ then the random variable, $Y = nln(\frac{X}{\beta})$ has the logbeta distribution with parameters $m, n\ and\ \alpha$. That is $Y \sim logbeta(m, \alpha - 1, \frac{\alpha-1}{n})$.

Using the transformation method, it can be easily proved that if a random variable

X ~ BLBP$(\alpha, \beta, m, n)$ then the random variable, $Y = nln(\frac{X}{\beta})$ has the logbeta distribution with the PDF of Y given as,

$f_Y(y) = \frac{\alpha-1}{nB(m,n)}[1 - e^{-(\frac{\alpha-1}{n})y}]^{(m-1)}e^{-(\alpha-1)y}$ (4.4)

In the above equation (4.4) if $m = 1$, is taken the PDF of the random variable Y becomes,

$f_Y(y) = \frac{\alpha-1}{\beta B(1,n)}\ e^{-(\alpha-1)y} = (\alpha - 1)e^{-(\alpha-1)y}$ (4.5)

Therefore, it follows from equation (4.5) that if random variable X ~ BLBP$(\alpha, \beta, 1, n)$ then the random variable $Y = nln(\frac{X}{\beta})$ follows the exponential distribution with mean $\frac{1}{k-1}$.

**Case 5:** If a random variable X ~ BLBP$(\alpha, \beta, m, n)$ then the random variable $Y = nln(\frac{X}{\beta})$ follows the Beta-Weibull distribution given that; $\alpha - 1 = n$, and the PDF of the random variable is given as,

$f_Y(y) = \frac{1}{B(m,n)}[1 - e^{-(m-1)y}]e^{-ny}$ (4.6)

Which is a special case of Beta-Weibull distribution.

## 5. Conclusion:

This paper discusses the genesis of Beta generated probability distributions. First this paper begins with the derivation of Length Biased Pareto Distribution and then using the beta generator technique a new probability distribution named Beta Length Biased Distribution is proposed. Various structural properties of the new distribution are considered in the paper. The new distribution is found to contain in it various other distributions under certain conditions.

**REFERENCES:**


1. Abramowitz, M. and Stegun, I.A. (1972). Handbook of Mathematical Functions (edited volume), Dover Publications, Inc., New York.
2. Akinsete, A., F. Famoye and C. Lee. (2008). The beta Pareto distribution, *Statistics*, 42, 547-563.
3. Ali, M.M. and Nadarajah, S. (2006). A truncated Pareto distribution, Comput. Commun., 30, 1-4.
4. Cox, D. R. (1969). Some sampling problems in technology, New Development in Survey Sampling, Wiley- Interscience, New York, 506-527.
5. Cordeiro, G. M., and De Castro, M. (2011). A new family of generalized distributions, J. of Stat. Comp. and Simulation, 81, 883-893.
6. Eugene, N., C. Lee and Famoye, F. (2002). The beta normal distribution and its applications. Commun. Statist. Theory Meth., 31, 497-512.
7. Jones, M. C. (2004). Families of distributions arising from distributions of order statistics, Test 13, 1-43.
8. Khattree, R. (1989). Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions. IEEE Trans. Reliab., 38: 610-611.
9. Lawless, J.F. (2011). Statistical Models and Methods for Lifetime Data. 2nd Edn., John Wiley and Sons, Hoboken, N.J.
10. Lee, E. T. and Wang, J. W. (2003). Statistical Methods for Survival Data Analysis. Wiley-Interscience, New Jersy.
11. Mahmoudi, E. (2011). The beta generalized Pareto distribution with application to lifetime data, *Math. Comp. Simul.*, 81, 2414-2430.
12. Nassar, M. M. and Nada, N. K. (2011). The beta generalized Pareto distribution, *J. Stat. Adv. Theory Appl., 6, 1-17* .
13. Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution, Math. Probab. Eng. (10), 323-332.
14. Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. Reliab. Eng. Syst. Safety, 91, 689-697.
15. Nanuwong, N. and Bodhisuvan, W. (2014). Length Biased Beta-Pareto Distribution and its Structural Properties with Applications. Journal of Mathematics and Statistics 10 (1), 49-57.
16. Newman, M.E.J. (2005). Power laws, Pareto distributions and Zipf's law. Contemporary Phy., 46, 323-351.
17. Patil, G. P. and Ord, J. K. (1976). On sized-biased sampling and related form-invariant weighted distributions, *Sankhya, Series B* 38(1), 48-61.
18. Patil, G.P. and C.R. Rao. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. Biometrics, 34: 179-189.
19. Rao, C. R. (1965). On discrete distributions arising out of methods of ascertainment, Statistical Publishing Society, Calcutta, 320-332.