REDUCTION OF DOUBLE HYPERGEOMETRIC SERIES

DR. MOHAMMAD SHAHJADE

Assistant Professor of Mathematics

MANUU, Polytechnic, 8th Cross, 1st Stage, 3rd Block Nagarbhavi, Bangalore-560072

MAULANA AZAD NATIONAL URDU UNIVERSITY, HYDERABAD-500032, INDIA

(CENTRAL UNIVERSITY)

E-mail: mohammadshahjade@gmail.com

Abstract: In this paperwork, we have taken certain transformation formulae due to Slater [2]; App. (III) Verma & Jain [1] and making use of known identities, to establish some double hyper geometric series into single series.

Key words: Generalized hyper - geometric function / Gauss hyper - geometric function and Ordinary hyper-geometric series; identities, known transformation formulae.

2010AMS Subject Classification: 33C20.

1. Introduction, Notation and Definitions

We shall make use of the following well known identities:

$$\sum_{n=0}^{n=\infty} \sum_{k=0}^{n} B(n,k) = \sum_{n=0}^{n=\infty} \sum_{k=0}^{\infty} B(k,n+k), \qquad (1.1)$$

An explicit representation of generalized hyper geometric functions

$$ {}_rF_s \left[\begin{matrix} a_1, & a_2, & \ldots, & a_r; & z \\ b_1, & b_2, & \ldots, & b_s \end{matrix} \right] = {}_rF_s \left[\begin{matrix} (a)_r; & z \\ (b)_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[(a_r)]_n \; z^n}{[1]_n \; [(b_s)]_n}, \qquad (1.2)$$

Valid for $|z| < 1$, provided no zeros appear in denominator. Here $a_1, a_2, a_3, \cdots, a_r$ and $b_1, b_2, b_3, \cdots, b_s$ and z are assumed to be complex number.

The shifted factorial is defined by

$$(a)_n = \left\{ \begin{matrix} 1, & n = 0 \\ a(a+1) \ldots\ldots\ldots\ldots\ldots\ldots (a+n+1); & n > 0 \end{matrix} \right\} \qquad (1.3)$$

In order to establish the reducibility of double hyper-geometric series into single series, we shall be need of the following known summation formulae due to (Slater [2], App.III) and Verma & Jain [1] in our analysis:

$$ {}_3F_2 \left[\begin{matrix} -n, & -n-x, & y; & 1 \\ & 1+x, & -n-y \end{matrix} \right] = \frac{(1)_n \, (1+x-y)_m (1+y)_m}{(1+x)_m \, (1+y)_n (1)_m}, \qquad (1.4)$$

where m is greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (2.7) P. 1024]

$$ {}_3F_2\left[\begin{matrix}-n, & -n-x, & 1+y; & 1\\ 1-n-x, & 1-n-y; & \end{matrix}\right] = \frac{(-)^n\,(1)_n\,(1+x+y)_n\,(1+x)_m\,(1+y)_m}{(x)_n\,(y)_n(1+x+y)_m(1)_m}, \tag{1.5} $$

Provided that m is the greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (2.11) P.1025]

$$ {}_3F_2\left[\begin{matrix}-n, & 1+n+2x+2y, & x; & 1\\ 1+x+y, & 1+2x; & & \end{matrix}\right] = \frac{(1)_n\,(1+x)_m\,(1+y)_m}{(1+2x)_n(1)_m\,(1+x+y)_m}, \tag{1.6} $$

where m is the greatest integer $\leq \frac{n}{2}$.

[Verma & Jain [1]; (2.26) P. 1028]

$$ {}_4F_3\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & -n; & 1\\ \frac{1}{2}a, & 1+a-b, & 2+2b-n; & & \end{matrix}\right] = \frac{(a-2b-1)_n(\frac{1}{2}a+\frac{1}{2}-b)_n\,(-b-1)_n}{(1+a-b)_n(\frac{1}{2}a-\frac{1}{2}-b)_n\,(-2b-1)_n}, \tag{1.7} $$

[Slater [2]; App. III (III.18)]

$$ {}_5F_4\left[\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & -n; & 1\\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+n; & & \end{matrix}\right] = \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n}, \tag{1.8} $$

[Slater [2]; App. III (III.13)]

2. Main Results:

In this section, we establish the following results,

2.1 Summation (1.4) can be written as:

$$ \sum_{k=0}^{n}\frac{(-n)_k\,(x)_k\,(y)_k}{k!(1+x)_k(-n-y)_k} = \frac{(1)_n\,(1+x-y)_n\,(1+y)_m}{(1+x)_m\,(1+y)_n\,(1)_m}, \tag{2.1.1} $$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.1.1), we have

$$ \sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{A_n Z^n\,(-n)_k\,(-n-x)_k\,(y)_k}{k!(1+x)_k(-n-y)_k} = \sum_{n=0}^{\infty}\frac{A_n Z^n\,(1)_n\,(1+x-y)_m\,(1+y)_m}{(1+x)_m\,(1+y)_n\,(1)_m}, \tag{2.1.2} $$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(1+y)_{n+k}}{(1+x)_{n+k}\,(1)_{n+k}}\,B_{n+k}$ and taking $B_n = 1$ in (2.1.2), we get:

$$ {}_1F_1[1+y;1+x;z]\times{}_1F_1[y,1+x;\,-z] = \sum_{n=0}^{\infty}\frac{Z^n\,(1+x-y)_n\,(1+y)_m}{(1)_m(1+x)_m\,(1+y)_n}, \tag{2.1.3} $$

Where m is the greatest integer $\leq \frac{n}{2}$.

2.2 Next, Summation (1.5) can be written as:

$$ \sum_{k=0}^{n}\frac{(-n)_k\,(-n-x)_k\,(1+y)_k}{k!(1+x)_k(1-n-y)_k} = \frac{(-)^n\,(1)_n(1+x+y)_n\,(1+x)_m\,(1+y)_m}{(x)_n\,(y)_n(1+x+y)_m\,(1)_m}, \tag{2.2.1} $$

Multiply both side by $A_n Z^n$ and summing over n from 0 to ∞ in (2.2.1), we have

$$ \sum_{n=0}^{n=\infty}\sum_{k=0}^{n}\frac{A_n Z^n\,(-n)_k\,(-n-2x)_k\,(y)_k}{k!(-n-x)_k(1+2y)_k} = \sum_{n=0}^{\infty}\frac{A_n Z^n\,(1)_n\,(1+x+y)_n\,(1+x)_m\,(1+y)_m}{(1+x)_n\,(1+2y)_n\,(1+x+y)_m\,(1)_m}, \tag{2.2.2} $$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(1+y)_{n+k}}{(1+x)_{n+k}\,(1)_n}B_{n+k}$ and taking $B_n = 1$ in (2.2.2), we get:

$$1^{F_1}[y;\ 1+x;\ z]\times 1^{F_1}[1+y;\ 1+x;\ -z]=\sum_{n=0}^{\infty}\frac{Z^n\,(1+x+y)_n\,(1+x)_m(1+y)_m}{(1+x)_n\,(x)_n\,(1+x+y)_m\,(1)_m},\qquad (2.2.3)$$

Where m is the greatest integer $\le \frac{n}{2}$

2.3 Further, Summation (1.6) can be written as:

$$\sum_{n=0}^{n=\infty}\frac{(-n)_k\,(1+n+2x+2y)_k\,(x)_k}{k!(1+x+y)_k(1+2x)_k}=\frac{(1)_n(1+x)_m\,(1+y)_m}{(1+2x)_n\,(1+x+y)_m\,(1)_m},\qquad (2.3.1)$$

Multiply both side by A_nZ^n and summing over n from 0 to ∞ in (2.3.1), we have:

$$\sum_{n=0}^{\infty}\ \sum_{n=0}^{n=\infty}\frac{A_nZ^n\,(-n)_k\,(1+n+2x+2y)_k\,(x)_k}{k!(1+x+y)_k(1+2x)_k}=\sum_{n=0}^{\infty}\frac{A_nZ^n\,(1)_n(1+x)_m\,(1+y)_m}{(1+2x)_n\,(1+x+y)_m\,(1)_m},\qquad (2.3.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(1+2x+2y)_{n+k}\ B_{n+k}}{(1)_{n+k}}$ and taking $B_{n+k}=1$ in (2.3.2), we get:

$$\sum_{k,n=0}^{n=\infty}\frac{Z^n\,(1+2x+2y)_{n+2k}\,(x)_k(-Z)^k}{k!\,n!(1+x+y)_k(1+2x)_k}=\sum_{n=0}^{\infty}\frac{Z^n\,(1+2x+2y)_n\,(1+x)_m\,(1+y)_m}{(1+2x)_n\,(1+x+y)_m\,(1)_m},\qquad (2.3.2)$$

Where m is the greatest integer $\le \frac{n}{2}$

2.4 Further, Summation (1.7) can be written as:

$$\sum_{k=0}^{k=n}\frac{(a)_k\,(1+\frac{a}{2})_k\,(b)_k\,(-n)_k}{k!(\frac{a}{2})_k(1+a-b)_k\,(2+2b-n)_k}=\frac{(a-2b)_n\,(\frac{1}{2}+\frac{a}{2}-b)_n\,(-1-b)_n}{(1+a-b)_n\,(-\frac{1}{2}+\frac{a}{2}-b)_n(-2b-1)_n},\qquad (2.4.1)$$

Multiply both side by A_nZ^n and summing over n from 0 to ∞ in (2.4.1), we have:

$$\sum_{n=0}^{n=\infty}\ \sum_{k=0}^{n}\frac{A_nZ^n\ (a)_k\,(1+\frac{a}{2})_k\,(b)_k(-n)_k}{k!(\frac{a}{2})_k(1+a-b)_k\,(2+2b-n)_k}=\sum_{n=0}^{\infty}\frac{A_nZ^n\,(a-2b)_n\,(\frac{1}{2}+\frac{a}{2}-b)_n\,(-1-b)_n}{(1+a-b)_n\,(-\frac{1}{2}+\frac{a}{2}-b)_n(-2b-1)_n},\qquad (2.4.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{(-2b-1)_{n+k}}{(1)_{n+k}}\ B_{n+k}$ and taking $B_{n+k}=\frac{(-\frac{1}{2}+\frac{a}{2}-b)_n}{(-b-1)_n}$ *and* $z=1$ in (2.4.2), we get:

$$\sum_{k,n=0}^{n=\infty}\frac{\left(-\frac{1}{2}+\frac{a}{2}-b\right)_{n+k}(a)_k\left(1+\frac{a}{2}\right)_k(-1-2b)_k}{k!\,n!(-1-b)_{n+k}\left(\frac{a}{2}\right)_k(1+a-b)_k}=\frac{\Gamma(1+a-b)\ \ \Gamma(\frac{3}{2}+2b-\frac{a}{2})}{\Gamma(2+b)\ \ \Gamma(\frac{1}{2}+\frac{a}{2})},\qquad (2.4.3)$$

Provided RL $(\frac{3}{2}+2b-\frac{a}{2})>0.$

2.5 Next, Summation (1.8) can be written as:

$$\sum_{k=0}^{k=n}\frac{(a)_k\,(1+\frac{a}{2})_k\,(b)_k\,(c)_k\,(-n)_k}{k!n!(\frac{a}{2})_k(1+a-b)_k\,(1+a-c)_k(1+a+n)_k}=\frac{(1+a)_n\,(1+a-b-c)_n}{(1+a-b)_n\,(1+a-c)_n},\qquad (2.5.1)$$

Multiply both side by A_nZ^n and summing over n from 0 to ∞ in (2.5.1), we have:

$$\sum_{n=0}^{n=\infty}\ \sum_{k=0}^{k=n}\frac{A_nZ^n\ (a)_k(1+\frac{a}{2})_k(b)_k(c)_k\,(-n)_k}{n!k!(\frac{a}{2})_k(1+a-b)_k\,(1+a-c)_k(1+a+n)_k}=\frac{A_nZ^n\ (1+a)_n\,(1+a-b-c)_n}{(1+a-b)_n\,(1+a-c)_n},\qquad (2.5.2)$$

Now applying the identity (1.1) and replacing A_{n+k} by $\frac{1}{(1+a)_{n+k}\ (1)_{n+k}}\ B_{n+k}$ and $B_n=(1+a-b)_n\,(1+a-c)_n$ in (2.5.2), we get:

$$\sum_{k=0}^{k=n}\frac{(1+a-b)_{n+k}\,(1+a-c)_{n+k}\ (a)_k\,(1+\frac{a}{2})_k\,(b)_k\,(c)_k\,(-n)_k}{n!k!(\frac{a}{2})_k(1+a-b)_k\,(1+a-c)_k(1+a)_{n+2k}}(-z)^k=(1-z)^{c+a-b-1}.\qquad (2.5.3)$$

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