THE BOHR– SOMMERFELD MODEL FOR ENERGY OF ELECTRON IN HYDROGEN ATOM

Dr.Motiur Rahman Ansari

Assistant Professor
Department of Physics,
Maulana Azad College of Engineering & Technology, Neora, Patna, Bihar, India, 801113.

Abstract: This paper discusses Bohr-Somerfield quantization theory and tries to explain how all earlier propositions about energy at atomic level failed to correspond to observed facts and further how this model solved the mathematical discrepancies leading to a near perfect model of energy exchange at the atomic level. The author has attempted here to show the historical connections, logical justifications as well as mathematical expressions for the energy of electron in hydrogen atom and prove how Bohr's celebrated theory of atomic structure is an application of Planck's theory of quanta to the Rutherford nuclear atom in an attempt, extraordinary fruitful, to define the nature of the orbits in which the electrons might revolve round the nucleus and to explain the origin of spectral lines of the elements. It further shows how Sommerfeld extended the model so as to include elliptical orbits. The introduction of the elliptic orbits has no new energy for the electron of the hydrogen atom. Bohr had also solved the dilemma arose due to the Rutherford's atomic model and he admitting the failure of the classical theory, applied with remarkable success of the quantum theory to the Rutherford nuclear atom with revolving electron, which leads us to understand the significance of the Bohr's atom model.

***IndexTerms* - Electromagnetic theory, Model, Electrical attraction, Energy, Atom.**

I. INTRODUCTION

The first real foundation of the modern concept of the atom was held by Faraday who discovered that in electrolysis each atom, irrespective of the nature of the element, gave up a fixed quantity of positive or negative charge. J.J.Thomson gave the first picture of the structure of the atom. He further made the assumption that the positive charge were uniformly distributed in a sphere of atomic dimension, a conception which seems to him most suited to mathematical treatment, while the electrons were so arranged inside the positive sphere that their mutual repulsion were exactly balanced by the force of attraction towards the centre of the mass.

In 1911, Rutherford explained successfully the structure of the atom on the basis of his experiment. He suggested that in an atom, the entire positive charge and nearly all of its mass are concentrated at the centre of the atom in a small volume known as the nucleus in planetary orbits at distance large compared with the size of the nucleus. In spite of strong experimental support he had faced certain difficulties. The revolving electrons are constantly accelerated towards the nucleus. Such electrons according to electromagnetic theory would constantly radiate energy in the form of electromagnetic waves. Hence they would rapidly spiral in and fall into the nucleus and the atom would collapse. In practice atom do not collapse. Rutherford also suggested that the electron might be assumed to revolve round the nucleus, like the planets round the sun at such a speed that the mechanical centrifugal force would just balance the next excess of electrostatic attraction and in consequence stability could secure.

Niel Bohr explained the Rutherford's atomic model on the basis of Planck's quantum hypothesis. He proposed that an electron can move only those orbits for which the angular momentum of the electron is an integral multiple of , where h is Planck's constant. The electron moving in any of the permitted orbits does not radiate energy in spite of its acceleration towards the centre of the orbit. The atom therefore is said to be exists in a stationary state. The emission or absorption of radiation by the atom takes place when an electron jumps from one permitted orbit to another. The radiation is emitted or absorbed as a single quantum whose energy is equal to the difference in energies of the electron in the two orbits involved.

In 1915 A. Sommerfeld the German physicist extended Bohr's theory by incorporating the idea of elliptical electronic orbits and taking into consideration the relativistic variation of electron mass. The atom proposed by Sommerfeld is therefore called the Sommerfeld relativistic atom model.

II. THEORY

In 1913, Niel Bohr removed the deficiencies of Rutherford model by incorporating some adhoc quantum hypothesis into it. Bohr explained the origin of the line spectra in general terms on the basis of two central ideas. One is the concept of photon and the other is the concept of energy level of atoms.

Bohr fused the quantum idea with purely mechanical model of Rutherford and introduced the following three revolutionary adhoc postulates:

(1) Electrons revolve around a nucleus only in certain special orbits called stationary orbits though an infinite number of orbits are mechanical allowed. While moving in the permitted orbits, electrons do not emit or absorb electromagnetic radiation though they are in accelerated motion. Hence the atom is stable.

(2) The allowed electron orbits are those for which the angular momentum is an integral multiple of ħ, where h is the Planck's constant. The angular momentum of the electron is $I\omega = mvr$.

Accordingly,

$$mvr = n\hbar \tag{1}$$

(3) The radiates energy only when the electron jumps from one of the upper allowed orbits with energy say E_2 to a lower allowed orbit with energy say E_1. The change in the energy during the transition is given by

$$E_2 - E_1 = h\nu \tag{2}$$

where ν is the frequency of the emitted electron.

Bohr obtained the values of the energies of various states of hydrogen atom by assuming that an electron of mass m, charge e and velocity v revolves in a circular orbit of radius r around nucleus of mass M and charge Ze, where Z is the atomic number of the element. For hydrogen atom Z=1 then E=e. For a dynamical stable orbit, the centripetal force experienced by the electron equal to the force of electrical attraction between the nucleus and electron. Thus,

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r}$$

$$\text{or,}\quad v^2 = \frac{e^2}{r\,m} \qquad (3)$$

Introducing the quantum condition for the orbit, (from postulate 2) we have

$$P_\varphi = I\omega = \frac{nh}{2\pi}$$

$$\text{But,}\quad I\omega = mr^2\omega = mrv$$

Therefore,

$$mrv = \frac{nh}{2\pi} \qquad (4)$$

or,

$$v = \frac{nh}{2\pi mr} \qquad (5)$$

Dividing equation (3) by equation (5), we get

$$v = \frac{1}{4\pi\epsilon_0}\frac{2\pi e^2}{nh} \qquad (6)$$

From equation (5) we have,

$$r = \frac{nh}{2\pi mv} \qquad (7)$$

Substituting the value of v from equation (6) we get

$$r = \frac{4\pi\epsilon_0\, n^2h^2}{4\pi^2 me^2} \qquad (8)$$

Thus, the radius r of the permitted orbit is directly proportional to n^2 since all other quantities are constant, where n is an integers, n= 1, 2, 3, ----.These integers are called the quantum numbers of the respective orbits.

The total energy W of the electronic system is equal to the sum of the kinetic and potential energies. That is

W = kinetic energy + potential energy

The kinetic energy of electronic system $= \frac{1}{2}mv^2 = \frac{1}{2}\text{m}\left(\frac{1}{4\pi\epsilon_0}\frac{2\pi e^2}{nh}\right)^2$

$= \frac{me^4}{8\epsilon_0^2 n^2h^2}$ [using equation (6)]

The potential energy of electronic system $= -\frac{e^2}{r} = -\frac{me^4}{4\epsilon_0^2 n^2h^2}$ [using equation (8)]

Therefore total energy of the electronic system is given by

$$W_n = \frac{me^4}{8\epsilon_0^2 n^2h^2} - \frac{me^4}{4\epsilon_0^2 n^2h^2} = -\frac{me^4}{8\epsilon_0^2 n^2h^2} \qquad (9)$$

Where W_n being the energy of the electron corresponding to the n^{th} orbit.

In the equation (9) since all quantities except n, are constants and the orbital energy is inversely proportional to the square of the quantum number of the orbit. Evidently for any one particular orbit the energy is constant. Which means that as long as the electron remains in that orbit it cannot lose energy by radiation in contradiction to the classical electromagnetic theory. The interpretation of the negative sign associated with the expression for the orbital energy is important. As n increases the absolute numerical value of the energy decreases, but on account of negative sign the actual energy increases. This means that the outer orbits have greater energy than the inner ones. The negative sign also leads to another important conception that the electron is bound to the nucleus but attractive forces so that energy must be supplied to the electron in order to separate it completely from the nucleus.

Sommerfeld extended the Bohr's theory by assuming the existence of elliptical orbit for the electron. He argued that since the electron is moving around and under the influence of a massive nucleus, like a planet round the central massive sun, it might describe elliptical orbits as well. Now an electron moving in an elliptical orbit has two degree of freedom and its position at any instant can be fixed in terms of polar coordinates r and $\boldsymbol{\varphi}$, where r is the radial distance of the electron from the nucleus at one of the foci of the ellipse and $\boldsymbol{\varphi}$ the vectorial angle which the radius vector makes with the major axis of the ellipse as shown in the Figure 1. Sommerfeld postulated that each of these degree of freedom must be quantized separately.

According to the Wilson Sommerfeld quantization rule the angular and radial momenta P_θ and P_r are given by,

$$\oint_0^{2\pi} P_\theta\, d\theta = \text{k h} \qquad \text{and} \qquad \oint P_r\, dr = n_r\, \text{h} \qquad (10)$$

k and n_r where are integers called respectively azimuthal and radial quantum numbers. By integrating over a complete revolution, it can be proved that

$$1 - \epsilon^2 = \frac{k^2}{(k+n_r)^2} \qquad (11)$$

where ϵ is the eccentricity of the ellipse. Now we can put in the equation (11)

$\text{k} + n_r = \text{n}$ (Since n=1,2,3........)

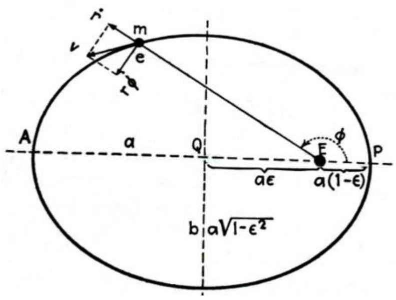

Figure 1

then we get ,

$$1-\epsilon^2 = \frac{k^2}{n^2} \tag{13}$$

Where n is the total quantum number of the electron.

If a and b be the semi – major and semi-minor axes of the ellipse then we have

$$1-\epsilon^2 = \frac{b^2}{a^2} \tag{14}$$

From equation (13) and equation (14) we have

$$\frac{b}{a} = \frac{k}{n} \tag{15}$$

This is the condition of quantization for the orbits. Only those elliptic orbits are permitted for the electron for which the ratio of the major to the minor axes is the ratio of two integers.

The total energy E of an electron in a quantized elliptical orbit is the sum of the kinetic energy K and the potential energy U. That is,

$$E = K + U$$

$$= \frac{1}{2}\,m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \tag{16}$$

But $P_r = m\dot{r}$ and $P_\theta = mr^2\dot{\theta}$ with this substitution equation (16) becomes,

$$E = \frac{1}{2}\,m\left({P_r}^2 + \frac{P_\theta^2}{r^2}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \tag{17}$$

Now $P_r = m\dot{r} = m\frac{dr}{dt} = m\,\frac{dr}{d\theta}\frac{d\theta}{dt} = m\dot{\theta}\,\frac{dr}{d\theta} = m\,r^2\dot{\theta}\,\frac{1}{r^2}\frac{dr}{d\theta} = \frac{P_\theta}{r^2}\frac{dr}{d\theta}$

Putting the value of P_r in the equation (17) we get,

$$E = \frac{P_\theta^2}{2mr^2}\left[\left(\frac{1}{r}\frac{dr}{d\theta}\right) + 1\right] - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \tag{18}$$

Now the polar equation of an ellipse is

$$\frac{1}{r} = \frac{1}{a}\,\frac{(1-\epsilon\cos\theta)}{1-\epsilon^2} \tag{19}$$

Differentiating equation (19), we get

$$\frac{1}{r^2}\frac{dr}{d\theta} = \frac{\epsilon\sin\theta}{a\,(1-\epsilon)^2} \tag{20}$$

Dividing equation (20) by equation (19) we get,

$$\frac{1}{r}\,\frac{dr}{d\theta} = \frac{\varepsilon\,\sin\theta}{1+\varepsilon\cos\theta} \tag{21}$$

Using equation (21) in equation (18) and solving them we get,

$$E = -\frac{me^4\,(1-e^2)}{(4\pi\epsilon_0)^2\;2\,P_\theta^2} \tag{22}$$

For an isolated system, the angular momentum P_θ is constant. Then from equation (1) we get,

$$P_\theta = \frac{kh}{2\pi}, \quad \text{also } 1-\epsilon^2 = \frac{b^2}{a^2} = \frac{k^2}{n^2}$$

Substituting these values in equation (22) we get,

$$E = -\frac{m\,e^4\left(\frac{k^2}{n^2}\right)}{(4\pi\epsilon_0)^2\;2\left(\frac{kh}{2\pi}\right)^2}$$

$$E = -\frac{m\,e^4}{8\,\epsilon_0^2\,h^2}\left(\frac{1}{n^2}\right) \tag{23}$$

This is the exactly the same as the energy of electron in a circular Bohr's orbit because E is independent of k and depends upon n only which is clear from equation (23). Thus more introduction of elliptic orbit adds no new energy level.

III. DISCUSSIONS

Bohr's celebrated theory of atomic structure is an application of Planck's theory of quanta to the Rutherford nuclear atom in an attempt, extraordinary fruitful, to define the nature of the orbits in which the electrons might revolve round the nucleus and to explain the origin of spectral lines of the elements. Sommerfeld extended the model so as to include elliptical orbits. The introduction of the elliptic orbits has no new energy for the electron of the hydrogen atom. Bohr had also solved the dilemma arose due to the Rutherford's atomic model and he admitting the failure of the classical theory, applied with remarkable success of the quantum theory to the Rutherford nuclear atom with revolving electron. This leads us to the consideration of the Bohr atom model.

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