# Static Cylindrically Symmetric Solutions in Einstein – Cartan Theory

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## Abstract

The present paper discussed the Einstein- Cartan field equations taking the static cylindrically symmetric perfect fluid distribution under different conditions. The constants appearing in the solution have been calculating using Licknerowicz boundary conditions.

## Key words

Symmetric, perfect fluid, cylindrical, Ricci tensor, Spin, Torsion Tensor, Licknerowicz boundary conditions.

**1 INTRODUCTION.**

In this study we discussed Einstein-Cartan theory which attempts to incorporate the spin density (or polarization) of a material medium into the field equations. Spherically symmetric interior solution in Einstein-Cartan theory were reported by Prasanna [15], Kerlick [6] and Kuchowicz [10-11] and Skinner and Webb [18]. Singh and Yadav [17] have also obtained static fluid spheres in Einstein-Cartan theory.Some other workers in this line are Suh [19] , Banerji [2], Arkuszewskiwski [1], Krori et .el.[8], Kopezynski [7]. However, since in spherical symmetry it is assumed that spins are all aligned in radial direction (implying the presence of a magnetic monopole at the centre) the picture is not very physical. Again, since a rotating system cannot be spherical, it is necessity to consider axisymmetric distributions which are more physical. In this connection Prasanna [16] has studied the simplest axisymmetric system namely a static cylinder of perfect fluid composed of particles having their spins aligned along the symmetric axis.

In this paper, we have solved the Einstein- Cartan field equations taking the static cylindrically symmetric perfect fluid distribution under different conditions. We have assumed the spins to be aligned along the symmetry axis. We have also evaluated pressure and density for the distribution. The constants appearing in the solution have been found using Licknerowicz boundary conditions.

## 2. THE FIELD EQUATIONS:

We take the static cylindrically symmetric metric given by

(2.1) $$ds^2 = -e^{2\mu-2\nu}\left(dr^2 + dz^2\right) - r^2 e^{-2\nu} d\phi^2 + e^{2\nu} dt^2,$$

where $\mu$ & $\nu$ are functions of r alone.

We have then the orthonormal tetrad

(2.2) $$\theta^1 = e^{\mu-\nu} dr, \theta^2 = re^{-\nu} d\phi, \theta^3 = e^{-\nu} dz, \theta^4 = e^{\nu} dt$$

The metric (2.1) now becomes,

(2.3) $$ds^2 = -\left\{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2\right\}$$

So that, $g_{ij} = diag(-1,-1,-1,1)$

The Einstein- Cartan field equations are

(2.4) $$R^i_j - \frac{1}{2} R\delta^i_j = -kt^i_j,$$

(2.5) $$Q^i_{jk} - S^i_j Q^l_{lk} - S^i_k Q^l_{lk} = -kS^i_{jk}$$

where $t^i_j$ is canonical asymmetric energy momentum tensor $R_{ij}$ is Ricci tensor, R is scalar of curvature, and $S^i_{jk}$ and $Q^i_{jk}$ are spin and torsion tensors.

The classical description of spin is defined by the relation,

(2.6) $$S^i_{jk} = u^i S_{jk} with S_{jk} u^k = 0$$

where $u^i$ is the velocity four vector and $S_{ij}$ is the intrinsic angular momentum tensor.

We suppose that spins of the particles composing the fluid area all aligned along the symmetry axis (Z-axis). Therefore the only non-zero components of spin tensor $S_{ij}$ are

(2.7) $$S_{12} = -S_{21} = K \text{ (say)}.$$

Also since the fluid distribution is static, the velocity four vector $u^i = \delta^i_4$ and hence the non-zero components of $S^i_{jk}$ are

(2.8) $$S^4_{12} == S^4_{21} = K$$

Therefore from Cartan's equation (2.5), we have the non- zero components of torsion tensor $Q^i_{jk}$ to be

(2.9) $$Q^4_{12} = -Q^4_{21} = -kK$$

For the system under study the canonical asymmetric energy momentum tensor $t^i_j$ is given by

(2.10) $$t^i_j = T^i_j + \frac{1}{2} g^{im} \nabla_K S^k_{jm}$$

$T_j^i$ being the symmetric energy momentum tensor. Considering the perfect fluid material distribution with anisotropic pressure, the symmetric tensor $T_j^i$ is given by

(2.11) $T_j^i = diag\{-p_r, p_\phi, p_z, \rho\}.$

Consequently the non- zero components of the canonical tensor $t_j^i$ are

(2.12)
$$t_1^1 = T_1^1 = -p_r, t_4^4 = T_4^4 = \rho$$
$$t_2^2 = T_2^2 = -p_\phi,$$
$$t_3^3 = T_3^3 = -p_z,$$
$$t_4^4 = T_4^4 = \rho,$$
$$t_2^4 = \frac{1}{2} K e^{\nu-\mu} \nu'$$
$$t_4^2 = \frac{1}{2} K e^{\nu-\mu} \nu'$$

From (2.4) & (2.12) the field equations may be written as

(2.13) $$e^{2(\nu-\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right) + \frac{1}{4}k^2K^2 = -k\rho$$

(2.14) $$e^{2(\nu-\mu)}\left(\nu^2 - \frac{\mu'}{r}\right) - \frac{1}{4}k^2K^2 = kp_r$$

(2.15) $$e^{2(\nu-\mu)}\left(-\mu'' - \nu'^2\right) - \frac{1}{4}k^2K^2 = kp_\phi,$$

(2.16) $$e^{2(\nu-\mu)}\left[-\nu'^2 + \frac{\mu'}{r}\right] - \frac{1}{4}k^2K^2 = kp_z,$$

(2.17) $$e^{2(\nu-\mu)}\left(K' + K\mu' - K\nu'\right) = -ke^{\nu-\mu}\nu'$$

(2.18) $$e^{\nu-\mu}\left(K' + K\mu' - K\nu'\right) = ke^{\nu-\mu}\nu'$$

Adding (2.17) & (2.18) we get,

(2.19) $K' + K\mu' = 0$ which on integration yields

(2.20) $K = A_1 e^{-\mu}$ where $A_1$ is an arbitrary constant to be determined. The conservations equation for j =1 gives the continuity equation.

(2.21) $$\frac{dp_r}{dr} + (\rho + p_r) - (p_r - p_\phi)\left(\nu' - \frac{1}{r}\right) - (\nu' - \mu')(p_r - p_z)$$
$$= -\frac{1}{2}kK(K' + K\nu')$$

It can be easily verified that the equation (2.21) may be obtained directly as a consequence of the field equations.

## 3 Solution of the field equations:

Following Hehl's [3,4] approached by redefining pressure and density as

(3.1) $$\overline{p} = p - 2\pi K^2, \overline{\rho} = \rho - 2\pi K^2$$

The field equations reduce to

(3.2) $$8\pi\overline{\rho} = e^{2(\nu-\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right)$$

(3.3) $$8\pi\overline{p_r} = -8\pi\overline{p_z} = e^2(\nu-\mu)\left(\frac{\mu'}{r} - \nu'^2\right)$$

(3.4) $$8\pi\overline{p_\phi} = e^{2(\nu-\mu)}\left(\mu'' - \nu'^2\right)$$

Also the continuity equation becomes

(3.5) $$\frac{d\overline{p_r}}{dr} + \left(\overline{\rho} + \overline{p_r}\right)\nu' - \left(\overline{p_r} - \overline{p_\phi}\right)\left(\nu' - \frac{1}{r}\right) - 2\overline{p_r}(\nu' - \mu') = 0$$

We have only three independent equations to determine five unknowns. Thus the system is indeterminate, we require two more conditions.

**Case I**

Here we assume an equation of state of the form

(3.6) $$\overline{\rho} = a\overline{p_r}$$

where 'a' is a constant. This gives an additional equation

(3.7) $$2\nu'' + \frac{2\nu'}{r} - (1-a)\nu'^2 = \frac{a\mu'}{r} + \mu''$$

Since our set of equation is still incomplete, we will take a suitable choice of one of the metric coefficients. For this we assume,

(3.8) $$\nu' = b_1 r^4 + b_2$$

were $b_1$ & $b_2$ are constants. With this value of $\nu$ ,equation (3.7) becomes

(3.9) $$\frac{d^2\mu}{dr^2} - \frac{a}{r}\frac{d\mu}{dr} = 30b_1 r^2 - (1-a)16{b_1}^2 r^6$$

Letting $\frac{d\mu}{dr} = p$, we have

(3.10) $$\frac{dp}{dr} + \frac{a}{r}p = 32b_1 r^2 - 16(1-a){b_1}^2 r^6$$

Equation (3.10) is a linear differential equation in $\rho$ &r. Its solution is given by

(3.11) $$p = \frac{d\mu}{dr} = c_1 r^{-a} + \frac{32b_1 r^3}{a+3} - \frac{16(1-a){b_1}^2 r^7}{a+7}$$

where $c_1$ is constant of integration. Integration of (3.11) gives

(3.12) $\mu = \frac{c_1 r^{1-a}}{1-a} + \frac{8b_1 r^4}{(3+a)} - \frac{2(1-a)b_1^2 r^8}{(a+7)} + c_2$

where $c_2$ is another constant of integration.

Now we have four arbitrary constants $b_1, b_2, c_1$ & $c_2$ which are to be determined by the boundary conditions .If we take $r=r_0$ to be radius of the cylinder, we have for $r>r_0$ (i.e for outside the cylinder ) the equation $R_{ij} = 0$ (empty space). A well known solution for Einstein equations for empty space with cylindrical symmetry is that given by Levi- Civita [21]which is given as

(3.13) $ds^2 = -Ar^{-2c(1-c)}\left(dr^2 + dz^2\right) - r^{2(1-c)}d\phi^2 + r^{2c}dt^2$

where c & A are constants. We use Licknerowicz boundary conditions, namely that metric potentials are continuous across the surface $r = r_0$ .Thus the continuity of $\mu, \mu'$ and $\nu, \nu'$

gives

(3.14) $b_1 = \frac{c}{4r_0^4}$

(3.15) $b_2 = c\log r_0 - \frac{c}{4}$

(3.16) $c_1 = \frac{r_0^{a-1}c}{4(1-a)}\left[\frac{4(1-a)c}{(a+7)r_0} - \frac{2(2a+9)}{a+3}\right]$

(3.17) $c_2 = \frac{1}{2}\log A + c^2\log r_0 - \frac{c^2}{(a+7)}r_0 + \frac{c(2a+9)}{2(1-a)(a+3)} + \frac{15c}{8(a+3)} + \frac{(1-a)}{7(a+7)r_0}^2 c^2$

Thus we have for the interior of the cylinder, the solution

$$\mu = c\left[\frac{c}{(a+7)r_0} - \frac{(2a+9)}{2(1-a)(a+3)}\right]\in^{1-a} + \frac{15c}{8(a+3)}\left(1+\in^4\right)$$

(3.18)

$$-\frac{c^2(1-a)}{8(a+7)r_0}\left(\in^8 - 1\right) + \frac{1}{2}\log A + c^2\log r_0 - \frac{c^2}{(a+7)r_0} + \frac{c(2a+9)}{2(1-a)(a+3)}$$

(3.19) $\nu = \frac{c}{4}\left(\in^4 - 1\right) + c\log r_0$

$where \in = \frac{r}{r_0}$

Also pressure and density are found to be

(3.20) $$8\pi\rho = 16\pi^2 H^2 e^{-2\mu} + G\left\{\begin{array}{l} 6\in^2 - a(a-1)\left[\frac{c}{a+7} - \frac{(2a+9)r_0}{2(1-a)(a+3)}\right]\in^{-(a+1)} \\ -\frac{45\in^2}{2(a+3)} + \frac{6c(1-a)\in^5}{(a+7)r_0} + 2\in^2 - c\in^6 \end{array}\right\}$$

(3.21) $$8\pi p_r = -8\pi p_z = 16\pi^2 H^2 e^{-2\mu} + G\left\{\begin{array}{l}(1-a)\left[\frac{c}{(a+7)r_0} - \frac{2a+9}{2(1-a)(a+3)}\right]\in^{-(a+1)} \\ + \frac{15\in^2}{2(a+3)} - \frac{c(1-a)\in^5}{(a+7)r_0} - c\in^6\end{array}\right\}$$

(3.22) $$8\pi p_\phi = 16\pi^2 H^2 e^{-2\mu} + G\left\{\begin{array}{l}a(a-1)\frac{c}{(a+7)r_0} - \frac{(2a+9)}{2(1-a)(a+3)}\in^{-(1+a)} \\ + \frac{45\in^2}{2(a+3)} - \frac{6c(1-a)}{(a+7)r_0}\in^5 + c\in^6\end{array}\right\}$$

where $G = \frac{ce^2(\nu-\mu)}{r_0{}^2}$

**CASE II**

Here we choose

(3.23) $\bar{\rho} = a\bar{p}_z$

Then we get an additional equation

(3.24) $2\nu'' + \frac{2\nu'}{r} - (1+a)\nu'^2 = \mu'' - \frac{a\mu'}{r}$

Also we assume $\nu$ same as in case I i.e.

(3.25) $\nu = b_1{}' r^4 + b_2{}'$

Using (3.25) equation (3.24) becomes

(3.26) $\frac{d^2\mu}{dr^2} - \frac{a}{r}\frac{d\mu}{dr} = 32b_1{}' r^2 - 16(1+a)b_1{}'^2 r^6$

Letting $\frac{d\mu}{dr} = p$, we get

(3.27) $\frac{dp}{dr} - \frac{a}{r}p = 32b_1{}' r^2 - 16(1+a)b_1{}'^2 r^6$

which is linear differential equation in p & r. Its solution is

(3.28) $p = c_1{}' r^a + \frac{32b_1{}'}{3-a}r^3 - \frac{16b_1{}'^2(1+a)r^7}{7-a}$

where $c_1{}'$ is constant of integration. Integration of (3.28) gives

(3.29) $\mu = c_1' \frac{r^{a+1}}{a+1} + \frac{8b_1' r^4}{3-a} - \frac{2b_1'^2 (1+a) r^8}{7-a}$

Where $c_2'$ is another constant of integration As in case I , using boundary conditions we can find the constants $b_1', b_2', c_1', c_2'$ and also pressure and density can be written similarly.

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