CFE FOR THE POWERS OF RATIO OF *k*-FIBONACCI NUMBERS

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Abstract

In this paper we determine some interesting identities related with sequence $\{F_{k,n}\}$ and use these identities to express the ration $\frac{F_{k,n}}{F_{k,n-r}}$ as finite continued fraction expansion. We defined $\phi_{F_{k,n}} = \lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-r}}$ and derive the closed from continued fraction expansion for $\phi^r_{F_{k,n}}$, for any positive integer *r*.

Keywords : Fibonacci number, Number theory, Fibonacci sequence and Positive integers etc.

Introduction :

The concept of the Common Front for Efficiency (CFE) in the context of the powers of ratios of k-Fibonacci numbers is a specialized topic in number theory, extending the classical Fibonacci sequence to a more general form. To provide a clear introduction, let's break it down systematically, focusing on the key components: k-Fibonacci numbers, their ratios, powers of these ratios, and the role of CFE.

We first obtain the extended Binet formula for $F_{k,n}$.

Theorem 1.1: $F_{k,n} = \dfrac{\left(\dfrac{k+\sqrt{k^2+4}}{2}\right)^n - \left(\dfrac{k-\sqrt{k^2+4}}{2}\right)^n}{\sqrt{k^2+4}} = \dfrac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$.

Proof: We prove the result by PMI. For $n = 1$, we have

$$F_{k,1} = \frac{\left(\dfrac{k+\sqrt{k^2+4}}{2}\right)^n - \left(\dfrac{k-\sqrt{k^2+4}}{2}\right)^n}{\sqrt{k^2+4}} = 1$$ which proves the result for $n = 1$.

Now, assume that result holds for all positive integers up to some positive integer m. Thus both $F_{k,m} = \dfrac{\alpha^m_{F_{k,n}} - \beta^m_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$ and $F_{k,m-1} = \dfrac{\alpha^{m-1}_{F_{k,n}} - \beta^{m-1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$ holds.

This gives, $F_{k,m} + F_{k,m-1}$

$$= k\left(\frac{\alpha_{F_{k,n}}^{m} - \beta_{F_{k,n}}^{m}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right) + \left(\frac{\alpha_{F_{k,n}}^{m-1} - \beta_{F_{k,n}}^{m-1}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}\right)$$

$$= \frac{k\left(\alpha_{F_{k,n}}^{m} - \beta_{F_{k,n}}^{m}\right) + \left(\alpha_{F_{k,n}}^{m-1} - \beta_{F_{k,n}}^{m-1}\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\alpha_{F_{k,n}}^{m-1}\left(k\alpha_{F_{k,n}} + 1\right) + \beta_{F_{k,n}}^{m-1}\left(k\beta_{F_{k,n}} + 1\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}.$$

Now we note that $k\alpha_{F_{k,n}} + 1 = k\frac{\left(k + \sqrt{k^2 + 4}\right)}{2} + 1 = \frac{k^2 + k\sqrt{k^2 + 4} + 2}{2}$

$$= \frac{k^2 + 2k\sqrt{k^2 + 4} + k^2 + 4}{4}$$

$$= \left(\frac{k + \sqrt{k^2 + 4}}{2}\right)^2 = \left(\alpha_{F_{k,n}}\right)^2.$$

Similarly it can be shown that $k\beta_{F_{k,n}} + 1 = \left(\beta_{F_{k,n}}\right)^2$.

Thus,

$$F_{k,m+1} = \frac{\alpha_{F_{k,n}}^{m-1}\left(k\alpha_{F_{k,n}} + 1\right)\beta_{F_{k,n}}^{m-1}\left(k\beta_{F_{k,n}} + 1\right)}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\left(\alpha_{F_{k,n}}\right)^{m-1}\left(\alpha_{F_{k,n}}\right)^2 - \left(\beta_{F_{k,n}}\right)^{m-1}\left(\beta_{F_{k,n}}\right)^2}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$= \frac{\left(\alpha_{F_{k,n}}\right)^{m+1} - \left(\beta_{F_{k,n}}\right)^{m+1}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

$$\therefore F_{k,m+1} = \frac{\left(\frac{k + \sqrt{k^2 + 4}}{2}\right)^{m+1} - \left(\frac{k - \sqrt{k^2 + 4}}{2}\right)^{m+1}}{\sqrt{k^2 + 4}} = \frac{\alpha_{F_{k,n}}^{m-1} - \beta_{F_{k,n}}^{m-1}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}$$

This proves the result for $n = m + 1$ and thus for all positive integers n.
We now obtain the limiting ratio of two consecutive k -Fibonacci numbers.

Lemma 1.2: $\phi_{F_{k,n}} = \lim_{x \to \infty} \frac{F_{k,n}}{F_{k,n-1}} = \frac{k + \sqrt{k^2 + 4}}{2}$.

Proof. We note that the sequence $\{x_n\}_{n=1}^{\infty} = \left\{\frac{F_{k,n}}{F_{k,n-1}}\right\}_{n=1}^{\infty}$ is convergent.

Let this sequence converges to some teal number x.

Now $\frac{F_{k,n+1}}{F_{k,n}} = k + \frac{F_{k,n-1}}{F_{k,n}}$. Then $\lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = k + \lim_{x\to\infty}\frac{F_{k,n-2}}{F_{k,n-1}} = k + \frac{1}{\lim_{x\to\infty}\frac{F_{k,n-1}}{F_{k,n-2}}}$. This gives x = k $+\frac{1}{x}$

$\Rightarrow x^2 - kx - 1 = 0$. Solving for x yields $x = \frac{k \pm \sqrt{k^2+4}}{2}$. Considering only the positive root, we get the required result.

Throughout the chapter, we denote $\alpha_{F_{k,n}} = \frac{k \pm \sqrt{k^2+4}}{2}$ and $\beta_{F_{k,n}} = \frac{k - \sqrt{k^2+4}}{2}$.

We now show how extended Binet formula for $F_{k,n}$ is useful to derive the above value of $\phi_{F_{k,n}}$

Lemma 1.3 :

$$\phi_{F_{k,n}} = \lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \frac{k = \sqrt{k^2+4}}{2}.$$

Proof. : Using extend Binet formula for $F_{k,n}$

We get,

$$\lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \lim_{x\to\infty}\frac{\dfrac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}}{\dfrac{\alpha^{n-1}_{F_{k,n}} - \beta^{n-1}_{F_{k,n}}}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}}}$$

$$= \lim_{x\to\infty}\frac{\alpha^n_{F_{k,n}} - \beta^n_{F_{k,n}}}{\alpha^{n-1}_{F_{k,n}} - \beta^{n-1}_{F_{k,n}}} = \lim_{x\to\infty}\frac{1 - \left(\dfrac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n}{\dfrac{1}{\alpha_{F_{k,n}}}\dfrac{1}{\beta_{F_{k,n}}}\left(\dfrac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n}$$

Now we note that $\left|\beta_{F_{k,n}}\right| < \alpha_{F_{k,n}}$. Also $\lim_{x\to\infty}\left(\frac{\beta_{F_{k,n}}}{\alpha_{F_{k,n}}}\right)^n \to 0$ (when n is sufficiently large).

Thus, $\lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-1}} = \frac{k + \sqrt{k^2+4}}{2}$, as required

Note :

1. $\alpha_{F_{k,n}} - \beta_{F_{k,n}} = \left(\frac{k + \sqrt{k^2+4}}{2}\right) - \left(\frac{k - \sqrt{k^2+4}}{2}\right) = \sqrt{k^2+4}.$

2. $\alpha_{F_{k,n}}\beta_{F_{k,n}} = \left(\frac{k + \sqrt{k^2+4}}{2}\right)\left(\frac{k - \sqrt{k^2+4}}{2}\right) = -1.$

Catalan's Identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan, a Belgian mathematician who worked for the Belgian Academy of Science in the field of Number Theory. Here we obtain analogues result.

Lemma 1.4

$$F_{k,n-r}F_{k,n+r}-F_{k,n}^2=(-1)^{n+1-r}F_{k,r}^2.$$

Proof. By using theorem 1.1 and $\alpha_{F_{k,n}}\beta_{F_{k,n}}=-1$ on LHS of result, we get

$$LHS=\frac{\left(\alpha_{F_{k,n}}^{n-r}-\beta_{F_{k,n}}^{n-r}\right)\left(\alpha_{F_{k,n}}^{n+r}-\beta_{F_{k,n}}^{n+r}\right)-\alpha_{F_{k,n}}^{2n}+2\alpha_{F_{k,n}}^{n}\beta_{F_{k,n}}^{n}-\beta_{F_{k,n}}^{2n}}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)}$$

$$=\frac{1}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left(-\alpha_{F_{k,n}}^{n+r}\beta_{F_{k,n}}^{n-r}-\alpha_{F_{k,n}}^{n-r}\beta_{F_{k,n}}^{n+r}+2\alpha_{F_{k,n}}^{n}\beta_{F_{k,n}}^{n}\right)$$

$$=\frac{1}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left(-\alpha_{F_{k,n}}^{n+2r-r}\beta_{F_{k,n}}^{n-r}-\alpha_{F_{k,n}}^{n-r}\beta_{F_{k,n}}^{n+2r-r}+2\alpha_{F_{k,n}}^{n}\beta_{F_{k,n}}^{n}\right)$$

$$=\frac{1}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[-\left(\alpha F_{k,n}\beta F_{k,n}\right)^{n-r}\alpha_{F_{k,n}}^{2r}-\left(\alpha F_{k,n}\beta F_{k,n}\right)^{n-r}\beta_{F_{k,n}}^{2r}+2\left(\alpha F_{k,n}\beta F_{k,n}\right)\right]$$

$$=\frac{1}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[-(-1)^{n-r}\alpha_{F_{k,n}}^{2r}-(-1)^{n-r}\beta_{F_{k,n}}^{2r}+2(-1)^{n}\right]$$

$$=\frac{1}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[(-1)^{n-r+1}\alpha_{F_{k,n}}^{2r}+(-1)^{n-r+1}\beta_{F_{k,n}}^{2r}+2(-1)^{n}\right]$$

$$=\frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[\alpha_{F_{k,n}}^{2r}+\beta_{F_{k,n}}^{2r}+2(-1)^{r-1}\right]$$

$$=\frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[\alpha_{F_{k,n}}^{2r}+\beta_{F_{k,n}}^{2r}+2(-1)^{r}\right]$$

$$=\frac{(-1)^{n-r+1}}{\left(\alpha_{F_{k,n}}-\beta_{F_{k,n}}\right)^2}\left[\alpha_{F_{k,n}}^{2r}+\beta_{F_{k,n}}^{2r}+2\left(\alpha_{F_{k,n}}+\beta_{F_{k,n}}\right)^{r}\right]$$

$$=(-1)^{n-r+1}\left(\frac{\alpha_{F_{k,n}}^{r}-\beta_{F_{k,n}}^{r}}{\alpha_{F_{k,n}}+\beta_{F_{k,n}}}\right)^2=(-1)^{n-r+1}F_{k,r}^2,$$ as required.

The following is analogues to one of the oldest identities involving the Fibonacci numbers, which was discovered in 1680 by Jean-Dominique Cassini, a French Astronomer.

Lemma 1.5

$$F_{k,n-1}F_{k,n+1}-F_{k,n}^2=(-1)^n. \qquad (1.1)$$

Proof. The proof follows immediately by taking $r = 1$ in Catalan's identity.

We next obtain extended d' Ocagne's identity

Lemma 1.6: $F_{k,n} = F_{k,n-r}F_{k,r+1} + F_{k,n-r-1}F_{k,r}$.

Proof. We write the extended Binet's formula from theorem 1.1 in the form

$$F_{k,n} = C_1^*\alpha_{F_{k,n}}^n + C_2^*\beta_{F_{k,n}}^n \qquad (1.2)$$

where $C_1^* = \dfrac{1}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}} = \dfrac{1}{\sqrt{k^2+4}}$ and $C_2^* = \dfrac{-1}{\alpha_{F_{k,n}} - \beta_{F_{k,n}}} = \dfrac{-1}{\sqrt{k^2+4}}$

We now express two consecutive terms of this sequence by using matrices as

$$\begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix} = \begin{bmatrix} \alpha_{F_{k,n}}^{n-r} & \beta_{F_{k,n}}^{n-r} \\ \alpha_{F_{k,n}}^{n-r-1} & \beta_{F_{k,n}}^{n-r-1} \end{bmatrix} \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix}. \text{ Then}$$

$$\begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} = \begin{bmatrix} \alpha_{F_{k,n}}^{n-r} & \beta_{F_{k,n}}^{n-r} \\ \alpha_{F_{k,n}}^{n-r-1} & \beta_{F_{k,n}}^{n-r-1} \end{bmatrix}^{-1} \begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix}$$

$$= \frac{1}{\alpha_{F_{k,n}}^{n-r}\beta_{F_{k,n}}^{n-r-1} - \alpha_{F_{k,n}}^{n-r-1}\beta_{F_{k,n}}^{n-r}} \begin{bmatrix} \beta_{F_{k,n}}^{n-r-1} & -\beta_{F_{k,n}}^{n-r} \\ -\alpha_{F_{k,n}}^{n-r-1} & \alpha_{F_{k,n}}^{n-r} \end{bmatrix} \begin{bmatrix} F_{k,n-r} \\ F_{k,n-r-1} \end{bmatrix}$$

$$= \frac{1}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} \begin{bmatrix} F_{k,n-r}\beta_{F_{k,n}}^{n-r-1} & -F_{k,n-r-1}\beta_{F_{k,n}}^{n-r} \\ -F_{k,n-r}\alpha_{F_{k,n}}^{n-r-1} & +F_{k,n-r-1}\alpha_{F_{k,n}}^{n-r} \end{bmatrix}$$

This gives

$$C_1^* = \frac{F_{k,n-r}\beta_{F_{k,n}}^{n-r-1} - F_{k,n-r-1}\beta_{F_{k,n}}^{n-r}}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}, C_2^* = \frac{-F_{k,n-r}\alpha_{F_{k,n}}^{n-r-1} + F_{k,n-r-1}\alpha_{F_{k,n}}^{n-r}}{\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

Thus $C_1^* = \dfrac{F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}}{\alpha_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}, C_2^* = \dfrac{-F_{k,n-r} + F_{k,n-r-1}\beta_{F_{k,n}}}{\beta_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$

Substituting values of C_1^* and C_2^* in (1.2), we get

$$F_{k,n} = C_1^*\alpha_{F_{k,n}}^n + C_2^*\beta_{F_{k,n}}^n$$

$$= \frac{\left(F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}\right)\alpha_{F_{k,n}}^n}{\alpha_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} + \frac{\left(-F_{k,n-r} + F_{k,n-r-1}\alpha_{F_{k,n}}\right)\beta_{F_{k,n}}^n}{\beta_{F_{k,n}}^{n-r-1}\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$= \frac{\alpha_{F_{k,n}}^{r+1}\left(F_{k,n-r} - F_{k,n-r-1}\beta_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)} + \frac{\beta_{F_{k,n}}^{r+1}\left(-F_{k,n-r} + F_{k,n-r-1}\alpha_{F_{k,n}}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$= \frac{F_{k,n-r}\left(\alpha_{F_{k,n}}^{r+1} - \beta_{F_{k,n}}^{r+1}\right) - \left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)F_{k,n-r-1}\left(\alpha_{F_{k,n}}^{r} - \beta_{F_{k,n}}^{r}\right)}{\left(\alpha_{F_{k,n}} - \beta_{F_{k,n}}\right)}$$

$$=F_{k,n-r}\left(\frac{\alpha_{F_{k,n}}^{r+1}-\beta_{F_{k,n}}^{r+1}}{\alpha_{F_{k,n}}-\beta_{F_{k,n}}}\right)-\left(\alpha_{F_{k,n}}\beta_{F_{k,n}}\right)F_{k,n-r-1}\left(\frac{\alpha_{F_{k,n}}^{r}-\beta_{F_{k,n}}^{r}}{\alpha_{F_{k,n}}-\beta_{F_{k,n}}}\right).$$

Now since $\alpha_{F_{k,n}}\beta_{F_{k,n}}=-1$, by using extend Binet formula for *k*- Fibonacci numbers, we get $F_{k,n}=F_{k,n-r}F_{k,n-r-1}F_{k,r.}$.

Theorem 1.7: $$\frac{F_{k,n}}{F_{k,n-r}}=\begin{cases}\left[F_{k,r+1}+F_{k,r-1};\frac{F_{k,n-r}}{F_{k,n-2r}}\right] & ;if\ r\ is\ odd\\ \left[F_{k,r+1}+F_{k,r-1}-1;1,\frac{F_{k,n-r}}{F_{k,n-2r}}-1\right] & ;if\ r\ is\ even\end{cases}$$

which yields

$$\phi_{F_{k,n+1}}^{r}=\begin{cases}\left[F_{k,r+1}+F_{k,r-1};\overline{F_{k,r+1}+F_{k,r-1}}\right] & ;if\ r\ is\ odd\\ \left[F_{k,r+1}+F_{k,r-1}-1;\overline{1,F_{k,r+1}+F_{k,r-1}+2}\right] & ;if\ r\ is\ even\end{cases}$$

Proof. By using lemma 1.6, we have

$$=F_{k,r+1}+\frac{F_{k,r}F_{k,n-1}}{F_{k,n-r}}$$

$$=F_{k,r+1}+\frac{kF_{k,r-1}F_{k,n-r-1}+F_{k,r-2}F_{k,n-r-1}}{F_{k,n-r}}$$

$$=F_{k,r+1}+\frac{kF_{k,r-1}F_{k,n-r-1}+F_{k,r-1}F_{k,n-r-2}+F_{k,r-2}F_{k,n-r-1}-F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}$$

$$=F_{k,r+1}+F_{k,r-1}\frac{F_{k,n-r}}{F_{k,n-r}}\frac{F_{k,r-2}F_{k,n-r-1}-F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}$$

$$=F_{k,r+1}+F_{k,r-1}+\frac{F_{k,r-2}F_{k,n-r-1}-F_{k,r-1}F_{k,n-r-2}}{F_{k,n-r}}$$

Again using lemma 1.6, we get

$F_{k,n-r-1}=F_{k,n-r-r}+F_{k,r-1}F_{k,n-r-r-1}$ and

$F_{k,n-r-2}=F_{k,r-1}F_{k,n-r-r-1}+F_{k,r-2}F_{k,n-r-r-2}$.

This gives

$$\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}$$

$$+\frac{\left(F_{k,r-2}\left(F_{k,r}F_{k,n-2r}+F_{k,r-1}F_{k,n-2r-1}\right)-F_{k,r-1}\left(F_{k,r-1}F_{k,n-2r}+F_{k,n-2}F_{k,n-2r-1}\right)\right)}{F_{k,n-r}}$$

$$=F_{k,r+1}+F_{k,r-1}+\frac{F_{k,r-2}\left(F_{k,r}F_{k,n-2r}\right)-F_{k,r-1}\left(F_{k,r-1}F_{k,n-2r}\right)}{F_{k,n-r}}$$

$$= F_{k,r+1} + F_{k,r-1} + \frac{F_{k,n-2r}\left\{F_{k,r-2}F_{k,r} - \left(F_{k,r-1}\right)^2\right\}}{F_{k,n-r}}.$$

Now using lemma 1.5, we get

$$\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + \frac{F_{k,n-2r}\left(-1\right)^{r-1}}{F_{k,n-r}}.$$

This gives $\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + F_{k,r-1} + \frac{\left(-1\right)^{r-1}}{F_{k,n-r}/F_{k,n-2r}}.$

Now if r is odd then we get

$$\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + F_{k,r-1} + \frac{1}{F_{k,n-r}/F_{k,n-2r}}.$$

$$\therefore \phi^r_{F_{k,n}} = \lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + \frac{1}{\lim\limits_{x\to\infty}\frac{F_{k,n-r}}{F_{k,n-2r}}}$$

$$= F_{k,r+1} + F_{k,r-1} + \frac{1}{\phi^r_{F_{k,n}}}$$

$$= F_{k,r+1} + F_{k,r-1} + \cfrac{1}{F_{k,r+1} + F_{k,r-1} + \cfrac{1}{F_{k,r+1} + F_{k,r-1} + \cfrac{1}{\ddots}}}.$$

Also if r is even then we get

$$\frac{F_{k,n}}{F_{k,n-r}} = F_{k,r+1} + F_{k,r-1} + \frac{-1}{F_{k,n-r}/F_{k,n-2r}}.$$

In this case we manipulate further. We write it as

$$\frac{F_{k,n}}{F_{k,n-r}} = \left(F_{k,r+1} + F_{k,r-1} - 1\right) + 1 - \frac{1}{F_{k,n-r}/F_{k,n-2r}}. \qquad (1.3)$$

Now $1 - \frac{1}{F_{k,n-r}/F_{k,n-2r}} = \frac{F_{k,n-r} - F_{k,n-2r}}{F_{k,n-r}}$

$$= \frac{1}{F_{k,n-r}/\left(F_{k,n-r} - F_{k,n-2r}\right)}$$

$$= \frac{1}{\left\{\left(F_{k,n-r} - F_{k,n-2r}\right) + F_{k,n-2r}\right\}/\left(F_{k,n-r} - F_{k,n-2r}\right)}$$

$$= \frac{1}{1 + \left\{\left(F_{k,n-2r}\right)/\left(F_{k,n-r} - F_{k,n-2r}\right)\right\}}$$

$$=\frac{1}{1+\left\{1/\left(\left(F_{k,n-r}-F_{k,n-2r}\right)/F_{k,n-2r}\right)\right\}}$$

$$=\frac{1}{1+\left\{1/\left(\left(F_{k,n-r}/F_{k,n-2r}\right)-1\right)\right\}}$$

$$\therefore \frac{F_{k,n}}{F_{k,n-r}}=\left(F_{k,r+1}+F_{k,r-1}-1\right)+\frac{1}{1+\left\{1/\left(\left(F_{k,n-r}/F_{k,n-2r}\right)-1\right)\right\}}.$$

Taking the limit as $n \to \infty$, we get

$$\phi^{r}_{F_{k,n}}=\lim_{x\to\infty}\frac{F_{k,n}}{F_{k,n-r}}=F_{k,r+1}+F_{k,r-1}-1+\frac{1}{1+\left(\dfrac{1}{\phi^{r}_{F_{k,n}}-1}\right)}$$

$$=F_{k,r+1}+F_{k,r-1}-1+\cfrac{1}{1+\cfrac{1}{F_{k,r+1}+F_{k,r-1}-2+\cfrac{1}{1+\cfrac{1}{F_{k,r+1}+F_{k,r-1}-2+\cfrac{1}{\ddots}}}}}$$

This completes the proof.

References :

[1] Feinberg M., “Fibonacci-Tribonacci”, *Fibonacci Quarterly*, Vol. l, No. l (1963), pp.70-74.

[2] Mccarty “A Formula for Tribonacci Sequence”, *Fibonacci Quarterly*, Vol.3 No.22 (1984), pp. 244-246.

[3] Parker F.D., “A Fibonacci Function” *The Fibonacci Quarterly* Vol.6 No. l (1968) pp. 1-2.

[4] Pethe S.P., Horadam’s A.F. “Generalized Gaussian Fibonacci Numbers”, *Bulletin Australian Math Society*, Vol.33, No. l (1986), pp. 37-48.

[5] Pethe S.P., Phadte C.N. “Generalization of the Fibonacci Sequence”, *Applications of Fibonacci Numbers* Kluwer Academic Publication Vol.5 (1993), pp. 465-472.

[6] Posamentier Alfred S. and Ingmar Lehmann “The Fabulous Fibonacci Numbers” *Prometheus Book*, 2007 edition.

[7] Spickerman W.R. “Binets Formula for the Tribonacci Sequence”, *Fibonacci Quarterly 20*, Vol.2 (1982), pp. 118-120.

[7] T. Koshy, Fibonacci and Lucas numbers with applications, Wiley Inter science, New York, 2001.