AN ANALYTICAL STUDY ON TWO BODY PROBLEM USING CELESTIAL MECHANICS

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ABSTRACT: *KAM theory: it provides the persistence of quasi–periodic motions under a small perturbation of an integrable system. KAM theory can be applied under quite general assumptions, i.e. a non– degeneracy of the integrable system and a Diophantine condition of the frequency of motion. It yields a constructive algorithm to evaluate the strength of the perturbation ensuring the existence of invariant tori. Perturbation theory: it provides an approximate solution of the equations of motion of a nearly–integrable system. Spin–orbit problem: a model composed by a rigid satellite rotating about an internal axis and orbiting around a central point–mass planet; a spin– orbit resonance means that the ratio between the revolutional and rotational periods is rational. Three–body problem: a system composed by three celestial bodies (e.g. Sun–planet–satellite) assumed as point–masses subject to the mutual gravitational attraction. The restricted three–body problem assumes that the mass of one of the bodies is so small that it can be neglected.*

KEYWORDS: KAM theory, Celestial Bodies, Kepler's laws, artificial Satellites , Taylor series, etc.

1. INTRODUCTION

Perturbation theory aims to find an approximate solution of nearly–integrable systems, namely systems which are composed by an integrable part and by a small perturbation. The key point of perturbation theory is the construction of a suitable canonical transformation which removes the perturbation to higher orders. A typical example of a nearly–integrable system is provided by a two–body model perturbed by the gravitational influence of a third body whose mass is much smaller than the mass of the central body. Indeed, the solution of the three–body problem greatly stimulated the development of perturbation theories. The solar system dynamics has always been a testing ground for such theories, whose applications range from the computation of the ephemerides of natural bodies to the development of the trajectories of artificial satellites.

The two–body problem can be solved by means of Kepler's laws, according to which for negative energies the point–mass planets move on ellipses with the Sun located in one of the two foci. The dynamics becomes extremely complicated when adding the gravitational influence of another body. Indeed Poincar´e showed ([12]) that the three–body problem does not admit a sufficient number of prime integrals which allow integrating the problem. Nevertheless a special attention deserves the so–called restricted three–body problem, namely when the mass of one of the three bodies is so small that its influence on the others can be neglected. In this case one can assume that the primaries move on Keplerian ellipses around their common bary center; if the mass of one of the primaries is much larger than the other (as it is the case in any Sun–planet sample), the motion of the minor body is governed by nearly–integrable equations, where the integrable part represents the interaction with the major body, while the perturbation is due to the influence of the other primary. A typical example is provided by the motion of an asteroid under the gravitational attraction of the Sun and Jupiter. The small body may be taken not to influence the motion of the primaries, which are assumed to move on elliptic trajectories. The dynamics of the asteroid is essentially driven by the Sun and perturbed by Jupiter, since the Jupiter–Sun mass–ratio amounts to about 10^{-3}. The solution of this kind of problem stimulated the work of the scientists, especially in the XVIII and XIX centuries.

Indeed, Lagrange, Laplace, Leverrier, Delaunay, Tisserand and Poincar´e developed perturbation theories which are at the basis of the study of the dynamics of celestial bodies, from the computation of the ephemerides to the recent advances in flight dynamics. For example, on the basis of perturbation theory Delaunay ([8]) developed a theory of the Moon, providing very refined ephemerides. Celestial Mechanics greatly motivated the advances of perturbation theories as witnessed by the discovery of Neptune: its position was theoretically predicted by John Adams and by Jean Urbain Leverrier on the basis of perturbative computations; following the suggestion provided by the theoretical investigations, Neptune was finally discovered on 23 September 1846 by the astronomer Johann Gottfried Galle.

2. CLASSICAL PERTURBATION THEORY

THE CLASSICAL THEORY

Consider a nearly–integrable Hamiltonian function of the form

$$H(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi})\ ,$$

$\underline{\varphi} \in$

where h and f are analytic functions of $\underline{I} \in V$ (V open set of R^n) and T^n (T^n is the standard n–dimensional torus), while ε > 0 is a small parameter which measures the strength of the perturbation. The aim of perturbation theory is to construct a canonical transformation, which allows removing the perturbation to higher orders in the perturbing parameter. To this end, let us look for a canonical change of variables (i.e., with simplistic

Jacobian matrix) $\mathcal{C} : (\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$**, such that the Hamiltonian (1) takes the form**

$$H'(\underline{I}',\underline{\varphi}') \ = \ H \circ \mathcal{C}(\underline{I},\underline{\varphi}) \ \equiv \ h'(\underline{I}') + \varepsilon^2 f'(\underline{I}',\underline{\varphi}')$$

where h´ and f´ denote the new unperturbed Hamiltonian and the new perturbing function. To achieve such result we need to proceed along the following steps: build a suitable canonical transformation close to the identity, perform a Taylor series expansion in the perturbing parameter, require that the unknown transformation removes the dependence on the angle variables up to second order terms, expand in Fourier series in order to get an explicit form of the canonical transformation.

The change of variables is defined by the equations

$$\underline{I} \ = \ \underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}',\underline{\varphi})}{\partial \underline{\varphi}}$$
$$\underline{\varphi}' \ = \ \underline{\varphi} + \varepsilon \frac{\partial \Phi(\underline{I}',\underline{\varphi})}{\partial \underline{I}'}\ ,$$

Where $\Phi(\underline{I}',\underline{\varphi})$ **is an unknown generating function, which is determined so that (1) takes**

$$f(\underline{I},\underline{\varphi}) = f_0(\underline{I}) + \tilde{f}(\underline{I},\underline{\varphi})$$

the form (2). Decompose the perturbing function as

where f0 is the average over the angle variables and ˜ f is the remainder function defined $\underline{\omega} = \underline{\omega}(\underline{I})$

through $\tilde{f}(\underline{I},\underline{\varphi}) \equiv f(\underline{I},\underline{\varphi}) - f_0(\underline{I})$**. Define the frequency vector as**

$$\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}}$$
.

The precession of the perihelion of Mercury

As an example of the implementation of classical perturbation theory we consider the computation of the precession of the perihelion in a (restricted, planar, circular) three–body model, taking as sample the planet Mercury. The computation requires the introduction of Delaunay action–angle variables, the definition of the three–body Hamiltonian, the expansion of the perturbing function and the implementation of classical perturbation theory.

3. RESONANT PERTURBATION THEORY

The Resonant Theory

Let us consider a Hamiltonian system with n degrees of freedom of the form

$$H(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi})$$

and let $\omega_j(\underline{I}) = \frac{\partial h(\underline{I})}{\partial I_j}\ (j = 1, ..., n)$ **be the frequencies of the motion, which we assume to satisfy** $\ell,\ \ell < n$ **, resonance relations of the form**

$$\underline{\omega} \cdot \underline{m}_k = 0 \qquad \text{for } k = 1, ..., \ell$$
,

for suitable rational independent integer vectors m1, ..., m`. A resonant perturbation theory can be implemented to eliminate the non–resonant terms. More precisely, the aim is to construct a canonical transformation C: $(\underline{I}, \underline{\varphi}) \rightarrow (\underline{J}', \underline{\vartheta}')$ **such that the transformed**

$$H'(\underline{J}', \underline{\vartheta}') = h'(\underline{J}', \vartheta'_1, ..., \vartheta'_\ell) + \varepsilon^2 f'(\underline{J}', \underline{\vartheta}')$$

Hamiltonian takes the form

where h´ depends only on the resonant angles $\vartheta'_1, ..., \vartheta'_\ell$**. To this end, let us first introduce the angles** $\underline{\vartheta} \in \mathbf{T}^n$ **as**

$$\begin{aligned} \vartheta_j &= \underline{m}_j \cdot \underline{\varphi} \qquad j = 1, ..., \ell \\ \vartheta_k &= \underline{m}_k \cdot \underline{\varphi} \qquad k = \ell + 1, ..., n \end{aligned}$$
,

where the first ℓ **angle variables are the resonant angles, while the latter** $n - \ell$ **−angle variables are defined as suitable linear combinations so to make the transformation canonical together with the following change of coordinates on the actions** $\underline{J} \in \mathbf{R}^n$ **:**

$$\begin{aligned} I_j &= \underline{m}_j \cdot \underline{J} \qquad j = 1, ..., \ell \\ I_k &= \underline{m}_k \cdot \underline{J} \qquad k = \ell + 1, ..., n \end{aligned}$$
.

Three–Body Resonance

We consider the three–body Hamiltonian with perturbing function $\underline{\omega} \equiv$ **and let be the frequency of motion.**

$$(\omega_\ell, \omega_g)$$

We assume that the frequency vector satisfies the resonance relation

$$\omega_\ell + 2\omega_g = 0$$

According to the theory described in the previous section we perform the canonical

$$\vartheta_1 = \ell + 2g \qquad J_1 = \frac{1}{2}G$$
$$\vartheta_2 = 2\ell \qquad J_2 = \frac{1}{2}L - \frac{1}{4}G$$

change of Variables

The precession of the equinoxes

An example of the application of the degenerate perturbation theory in Celestial Mechanics is provided by the computation of the precession of the equinoxes.

We consider a triaxial rigid body moving in the gravitational field of a primary body. We introduce the following reference frames with common origin in the barycenter of the rigid body: is an inertial reference frame,

$$(O, \underline{i}_1^{(i)}, \underline{i}_2^{(i)}, \underline{i}_3^{(i)})$$

$$(O, \underline{i}_1^{(s)}, \underline{i}_2^{(s)}, \underline{i}_3^{(s)})$$

$$(O, \underline{i}_1^{(b)}, \underline{i}_2^{(b)}, \underline{i}_3^{(b)})$$

is a body frame oriented along the direction of the principal axes of the ellipsoid, is the spin reference frame with the vertical axis along the direction of the angular momentum, (J, g, ℓ)

Let be the Euler angles formed by the body and spin frames, and let (K, h, 0) be the Euler angles formed by the spin and inertial frames. The angle K is the obliquity (representing the angle between the spin and inertial vertical axes), while J is the non–principal rotation angle (representing the angle between the spin and body vertical axes).

4. PLANETARY PROBLEMS

The dynamics of the planetary problem composed by the Sun, Jupiter and Saturn is investigated. In the secular dynamics of the following model is studied: after the Jacobi's reduction of the nodes, the 4–dimensional Hamiltonian is averaged over the fast angles and its series expansion is considered up to the second order in the masses. This procedure provides a Hamiltonian function with two degrees of freedom, describing the slow motion of the parameters characterizing the Keplerian approximation (i.e., the eccentricities and the arguments of perihelion). Afterwards, action–angle coordinates are introduced and a partial Birkhoff normalization is performed. Finally, a computer–assisted implementation of a KAM theorem yields the existence of two invariant tori bounding the secular motions of Jupiter and Saturn for the observed values of the parameters.

The approach sketched above is extended in so to include the description of the fast variables, like the semi–major axes and the mean longitudes of the planets. Indeed, the preliminary average on the fast angles is now performed without eliminating the terms with degree greater or equal than two with respect to the fast actions. The canonical transformations involving the secular coordinates can be adapted to produce a good initial approximation of an invariant torus for the reduced Hamiltonian of the three–body

planetary problem. This is the starting point of the procedure for constructing the Kolmogorov's normal form which is numerically shown to be convergent. In the same result of has been obtained for a fictitious planetary solar system composed by two planets with masses equal to 1/10 of those of Jupiter and Saturn.

5. PERIODIC ORBITS

Construction of periodic orbits

One of the most intriguing conjectures of Poincar´e concerns the pivotal role of the periodic orbits in the study of the dynamics; more precisely, he states that given a particular solution of Hamilton's equations one can always find a periodic solution (possibly with very long period) such that the difference between the two solutions is small

$$\dot{x} = y$$

for an arbitrary long time. The literature on periodic orbits is extremely wide (see, e.g., [3], [7], [10], [14], [15] and references therein); here we present the construction of periodic orbits implementing a perturbative approach (see [20]) as shown by Poincar´e in [12]. We describe such method taking as example the spin–orbit Hamiltonian (23) that we write in a compact form as for a suitable function f = f(x, t); the corresponding Hamilton's equations are

$$H(y,x,t) \equiv \frac{y^2}{2} - \varepsilon f(x,t)$$

6. FUTURE DIRECTIONS

The end of the XX century has been greatly marked by astronomical discoveries, which changed the shape of the solar system as well as of the entourage of other stars. In particular, the detection of many small bodies beyond the orbit of Neptune has moved forward the edge of the solar system and it has increased the number of its population. Hundreds objects have been observed to move in a ring beyond Neptune, thus forming the so–called Kuiper's belt. Its components show a great variety of behaviors, like resonance clustering's, regular orbits, scattered trajectories. Furthermore, far outside the solar system, the astronomical observations of extra solar planetary systems have opened new scenarios with a great variety of dynamical behaviors. In these contexts classical and resonant perturbation theories will deeply contribute to provide a fundamental insight of the dynamics and will play a prominent role in explaining the different configurations observed within the Kuiper's belt as well as within extra solar planetary systems.

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