Using Mathematical Applications to Solve Physical Problems - An Empirical Study

*Dr.Shivaraj Gadigeppa Gurikar. Asst Professor of Physics. Govt First Grade College, Yelburga.

Abstract

This paper looks at Mathematics as an essential element of physics problem solving, which is often failed to appreciate usage of it. Pure mathematics and physics are becoming ever more closely connected. Dirac went on to say that the two subjects might unify, with “every branch of pure mathematics then having its physical application”. Dirac’s prognosis was, and remains, highly speculative. Today, there is no question of a unification of these fields. Techniques from pure mathematics are used in economics, engineering and finance, but there’s no sense in — nor reason for — these fields becoming one.The ability to precisely measure resistance comes from von Klitzing’s discovery that resistance is quantized at values that are proportional to a combination of two fundamental physical constants: the charge of the electron and Planck’s constant. Moreover, the value of the quantized resistance is accurate even when materials contain impurities, which would otherwise change the resistance. Because of this, the quantum Hall effect is used to confirm the accuracy of the ohm, the unit of electrical resistance. Von Klitzing received the Nobel Prize in Physics for this discovery in 1985, five years after his paper was published. Dirac’s sentiment rankles with pure mathematicians because it suggests that physicists regard mathematics more as a tool with which to study the natural world than as a discipline in its own right. Such a view can be a barrier to fruitful collaboration. But when mathematicians and physicists do attempt to solve problems on equal terms, the results can be sublime — as we have seen in the physics of materials and in topology, a branch of pure mathematics that studies shapes and how they are arranged in space.

Mathematicians and physicists working in these fields have made lasting contributions to understanding the quantum Hall effect, which was discovered during a transformative experiment 40 years ago1,2. How they achieved this holds lessons for the way in which disciplines — and not only those in the physical sciences — could more successfully engage with each other on common problems. The quantum Hall effect describes the process through which electrical resistance can be precisely measured in layers of material a few atoms thick. How has this come about? Is it good for mathematics? Is it good for physics? To understand the current situation one needs also to understand the differing aims and methods of both groups. A physicist’s attempt to understand physical reality is based on experiments, measurements and the recognition and formulation of laws.

Key words: Mathematicians, physicists, Dirac, , quantum Hall effect, experiments, measurements.

Introduction

One of the chief tools in physics is mathematics. As it turns out, the world is ordered such that we can apply mathematical rigor to our understanding of it. Thus, we will focus on how mathematical principles and techniques can be used in physics to solve various problems and to model physical phenomena. To be sure, the topic of math in physics could span numerous courses; as such, we will focus on some basic principles that rely on algebra, trigonometry, and geometry. The original Hall effect, discovered in 1879 by physicist Edwin Hall, describes how a magnetic field applied perpendicularly to a metal strip causes electrons to gather along both ends of the strip, creating a voltage. A century later, the physicist Klaus von Klitzing went further. Working at low temperatures with atomically thin layers of crystalline materials — known as two-dimensional electron systems — he discovered that this voltage is quantized. That is, the voltage changes in jumps, as the applied magnetic field changes. This phenomenon is the quantum Hall effect. Vector methods have become standard tools for the physicists. Physics is the study of the characteristics and interactions of matter and energy in nature.

By a single number (called its magnitude) such as volume, mass, and temperature is called a scalar. Scalar quantities are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on. There are also physical quantities which require a magnitude and a direction for their complete speciation. These are called vectors if their combination with each other is commutative (that is the order of addition may be changed without ejecting the result). Thus not all quantities possessing magnitude and direction are vectors. Angular displacement, for example, may be characterised by magnitude and direction but is not a vector, for the addition of two or more angular displacements is not, in general, commutative. But what of pure mathematics, and how did topology become involved? It turns out that, at the time, physics was unable to fully explain why the resistance changes in discrete steps when the magnetic field changes. Two years after von Klitzing's discovery, physicist David Thouless provided an explanation using topology4. His work was subsequently built on by others, and, in 2014, he was awarded a share of the Nobel Prize in Physics. But some mathematicians were not satisfied with the standard of proof offered by the physicists, and quantum Hall resistance was added to a famous list of unsolved problems in mathematical physics. It was not until 2014 — 33 years after Thouless's calculation — that a more-rigorous mathematical proof was published5 by the mathematician Spyridon Michalakis. The pair began working on the problem in 2008, as Michalakis wrote in Nature Reviews Physics earlier this month6. Theoretical physicists and mathematicians knew that the average curvature of a geometric object — such as its surface — has a topological nature. They also knew that small local deformations affect the curvature locally. But a more rigorous explanation of quantized Hall resistance needed the theory to extend to global curvature. This is what Michalakis and Hastings achieved, making the link between topology and the quantum Hall effect ironclad.

And the story isn't over, by any means. Topology has been getting more attention from physicists, and from funders such as the Simons Foundation in New York City, which is supporting mathematicians and physicists working on difficult problems, such as the fractional quantum Hall effect. In this phenomenon, complex electron interactions cause the Hall resistance to be quantized at a value that is just a fraction of the charge of the electron.

Rather than seek to unify the two disciplines, as Dirac proposed, perhaps the greatest incentive that physicists can create for mathematicians is to leave a problem partially solved. Ultimately, the mathematical proof for quantum Hall resistance might well not have come about had the question not been classified as one of mathematical physics' unsolved problems. According to Galileo Galilei, "Mathematics is the language with which God has written the universe," a view echoed 400 years later in Eugene Wigner's paper entitled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences". The past 30 years has seen a significant change, however, which some have characterized as "The Unreasonable Effectiveness of Physics in Mathematics".

Objective:

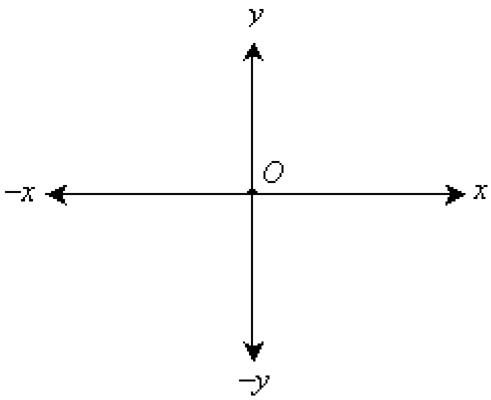
This paper intends to explore Math as the may be the language of physics, with focus on math-in-physics is a distinct dialect of that language. Which Physicists tend to blend conceptually

Locations of Objects and Events

To frame those rules, mathematics is necessary, but however sophisticated a tool, it is used for the purpose of better understanding the physical processes. Its ultimate validation is its agreement with experimentation, when that is possible.Simple graphical (coordinate) method of representing the locations of objects and events. These simple mathematical tools will provide us with a foundation on which we can build a system for analyzing motion, forces, energy, and other physical phenomena.

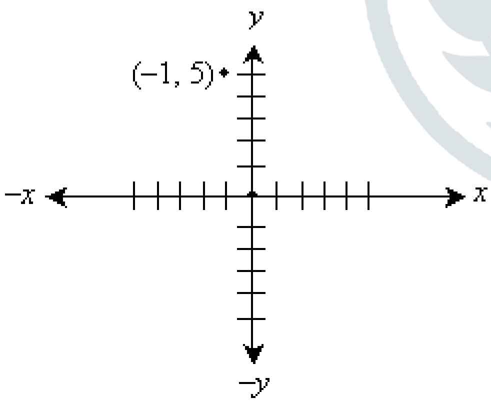
Physical objects and events have a spatial extent or location. As a result, it is helpful to have an orderly way in which we can describe these characteristics mathematically. One way to describe the position (location) of, for instance, a particle is to use a set of mutually perpendicular axes, just as we might do when graphing a function $y(x)$. Each axis corresponds to a direction (and its opposite), such as forward and backward or left and right. Each direction is mutually perpendicular with the other directions. (Obviously, if we are talking about three-dimensional space, which is largely how we perceive things and events around us, then we need only talk about three mutually perpendicular directions--up and down, left and right, and forward and backward, for instance.) In this course, we will deal primarily with objects and events in two dimensions for simplicity. The techniques and principles that we study, however, can easily (in most cases) be extended to three dimensions.

In addition to defining the mutually perpendicular dimensions for our system of identifying position in space, we also need to define a central point, or origin, that marks the spot from which we measure distances in each direction. A set of directions, or axes (marked as positive and negative x and y) and corresponding origin (point O) are shown below.



This system of locating an object or event might be as simple as a map where a city marks the origin, and the locations of other cities are noted as distances from the origin city in the directions north, south, east, or west. The choice of a set of directions and an origin is arbitrary as long as the axes (directions) are mutually perpendicular and span the proper space (the plane of interest, in the case of two dimensions--a map, for example, deals with directions in the plane of the Earth's surface). A set of axes and corresponding origin is also typically called a **frame of reference** (or **reference frame**) in the parlance of physics.

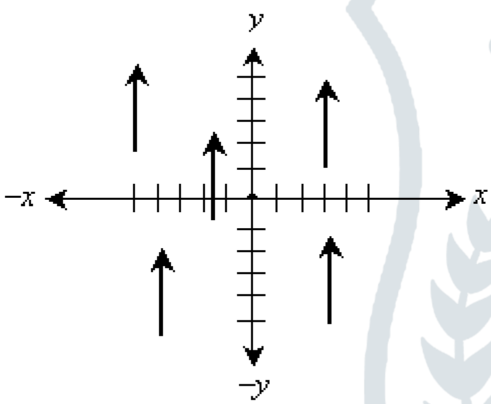
A location can be noted in two dimensions as a pair of coordinates of the form (*x*, *y*). Using standard algebraic graphing techniques, an object located at (–1, 5), for instance, could be shown as below.



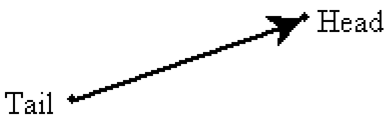
Vectors

In addition to identifying the location of a particular object or event, we may also want to quantify some other physical characteristic, such as temperature or velocity. In some cases, all we need is a number; for instance, we can talk about the temperature of an object by simply referring to a single number (and associated unit), such as 48 degrees Fahrenheit. This number is simply a magnitude that quantifies the physical characteristic--temperature, in the case of this example. In other cases, a number is not sufficient. The speed of the wind is helpful information, but it is not complete; in addition to a speed such as 20 miles per hour, wind also has a direction such as south or northeast. We therefore need more than just a simple number (called a **scalar**) to quantify characteristics such as velocity or force: we need to quantify direction also. For this purpose, we define a vector**,** which is a quantity with both a magnitude and a direction.

Graphically, we can show a direction using an arrow; we can also show a magnitude by the length of the arrow. An example of a vector with length of four units and directed in the positive *y* direction is shown below. Note that a vector has magnitude and direction but *not* location. For instance, imagine a wind of 40 miles per hour in the eastward direction. Whether such a wind blows in one place or another, it still has the same magnitude and direction. Likewise, a vector with a given magnitude and direction is the same regardless of its location. As a result, each vector shown in the graph below is identical because each has the same magnitude (four units) and direction (positive *y*).



The graphical form of a vector has two essential parts: the head (the endpoint corresponding to the arrow) and the tail (the endpoint opposite the head).



We would like to be able to assign a vector a simpler numerical designation that does not require us to specify magnitude and direction separately. Because a vector has no particular location, we can place the tail on the origin of our graph; thus, the tail is located at point (0, 0). Thus, only the head has a location whose coordinates are non-zero. To perform this relocation of the vector representation, we can simply subtract the tail coordinates from both the head coordinates and tail coordinates. This translates the vector such that the tail is at (0, 0), or the origin.

CURRENT INTERACTIONS : Core Methodical tools for physics

1) Vector Calculus: Even experimentalists need to know the basics of integration and differentiation in multiple dimensions. Need to understand gradient and curl and related operations on vector fields, and have a solid conceptual understanding of what it means to integrate along a path, over a surface, or through a full volume. If nothing else, if you hold out hope of an academic job, you'l need to teach this stuff someday.

2) Basic Differential Equations: There's a lot of truth to that--a huge range of problems can be made to look like small variations on the harmonic oscillator, so we spend a lot of time on that. The harmonic oscillator is one of the handful of differential equations with nice, friendly, easy-to-work-with solutions, and anybody working in physics needs to know how to work with all of those. And also the general technique for working with differential equations outside that handful, which boil down to "find a way to make it look like a perturbation on one of the equations we do know how to solve."

3) Basic Linear Algebra: The most compact and elegant expression of quantum mechanics is written in the language of linear algebra: vectors, matrices, eigenvalue problems, etc. Language from linear algebra even permeates the wave-mechanics versions of quantum mechanics, which can be a little confusing for students who haven't seen the math yet. It's absolutely essential to get this stuff down, because there's no getting away from it.

4) Basic Statistics: Stats are obviously essential for experimentalists who need to quantify the uncertainty in their measurements, but even theory has uncertainty, thanks to the need to put in experimental parameters. Anybody working in physics will need to have some understanding of standard deviations, error propagation, averaging techniques, etc. This material is also incredibly useful for understanding lots of public policy debates, so it's a win-win: it makes you a better physicist, and also a better citizen.

More recently, and certainly at the International Congress in Madrid, one experienced the shift from deterministic to stochastic methods, which have their origins in the 19th century physicists' study of thermodynamics. These movements sometimes originate from developments within the subject, sometimes from external influences. Both communities of mathematicians and physicists are alive and evolving, and aiming at discovery, but their backgrounds and motivations differ. This diversity is a source of strength if the two groups can focus on a common problem.

Perhaps the most exciting interaction between physics and mathematics at the moment is in Quantum Field Theory and String Theory. Any interaction is a two-way process but in the past few years it is the predictive power of String Theory in pure mathematics that is the most astonishing feature.

String Theory as manifestation of Mathematical construct

New facts and coincidences are being pointed out, not only in traditional areas with a common interface, but also way beyond that, in algebraic geometry and number theory. It is as if today's theoretical physics has had the power to jump into the interior of pure mathematics and tear it apart. The cynical might say that this is not an achievement of String Theory but a manifestation of its failure to be predictive about actual physical reality. Is it really a physical theory, or simply a set of analogies? Are mathematicians just feeding off the physicists' intuition because pure mathematics is the only place where the theory is applicable? The counterargument is to assert that String Theory is a consistent theory but it is so complicated that it has to use every tool in the mathematician's cupboard. It may still be true that "nature is the realization of the simplest possible mathematical ideas", it's just that you need to know a lot of mathematics to see how simple it is. String theorists would freely admit that they don't know what the theory is, but they are fairly sure that what they have is a genuine theory. What they observe is its implications at different limits of coupling constants where it makes contact with other areas of mathematics. The fundamental concepts in the terra incognita at its centre are unknown yet its deep consistency unearths structures across a wide range of mathematics. They also admit that is harder than they thought when the possibilities opened up in the mid 1980s, but by being harder it has drawn them closer to mathematics and they are quite happy to use the predictive power within that domain, given that the physical experiments are currently impractical.

Most mathematicians welcome this interaction and are happy to use the "unreasonable effectiveness of the equations of mathematical physics in pure mathematics". These have a history, since before String Theory. Hermann Weyl investigated the representation theory of groups because of its use in quantum mechanics, but it is now a tool throughout mathematics: in algebra, geometry and number theory. The more recent interactions involve the Fields Medal-winning work of Simon Donaldson using Yang-Mills equations to probe the topology of four-dimensional manifolds and that of Vaughan Jones and Edward Witten in defining knot invariants. These are practical (in a mathematical sense) theories that can be put to use in many areas, but often the crucial advances were achieved by pursuing the physicist's intuition. Some would say that these are advances that perhaps mathematicians did not deserve. In the current phase of interaction, mathematicians are now becoming familiar with the physicists' way of wrapping up mathematical information in a partition function. This becomes a formal means of counting objects that have been considered individually in the past but not systematically in such a way.

These objects might be algebraic curves, or numbers of intersections or numbers of solutions to certain equations, all wrapped up in a generating function. Sometimes there is a subtlety in counting multiplicities which has eluded the mathematicians but which is natural for the physicists and leads to functional equations which the generating functions satisfy. Given this global view, one can highlight certain examples. Donaldson theory is viewed as perturbative. The distinct, non-perturbative picture of the same theory yields Seiberg-Witten theory.

What the mathematicians found was that it gave them a brand-new method to prove efficiently precisely the results they were finding it hard to achieve with standard Donaldson theory. There was a period in 1995 when geometers burnt the midnight oil to race each other to proofs of some longstanding conjectures using this new method.

Conclusion

physicists believe in quantum field theory not because it is a rigorous piece of mathematics, but because it gives them the correct answers to many decimal places. They work in different ways from mathematicians, attacking current problems with a huge concentration of forces. They have no time to wait for the full mathematical theory but proceed with great momentum that carries them beyond the stage where the hypotheses are testable. Contrast this with the mathematician, willing to wait years to complete a theory, like Andrew Wiles's celebrated proof of Fermat's theorem or, closer to physical reality, Carl Friedrich Gauss's 25 years of secretly studying the differential geometry of surfaces (the physical reality that Gauss was attempting to describe there was founded in geodesy). The pure mathematician is, in the public's view, a practitioner of an art which "possesses not only truth, but supreme beauty –a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature", in Bertrand Russell's words. Can there be any common ground for both communities to work in? One answer is to say that science is not actually describing physical reality but is concerned with human understanding of it. In this view, the beauty and elegance of mathematics is a guide towards a theory that has a coherence and simplicity that aids our comprehension of nature. But beauty alone can lead the physicist astray.

Who can deny that Johannes Kepler's original view of the solar system based on the Platonic solids was beautiful? But it was plain wrong. Kepler tried hard to avoid the conclusion that the planetary orbits were elliptic but in the end he had to admit it was a fact. Whether he appreciated it as a manifestation of another beautiful piece of mathematics is not clear but what actually happened was that one elegant model was replaced by another more sophisticated one. It is perfectly possible to change one's view of what constitutes a simple elegant theory. The cause need not even be a failure of the theory to agree with experiment. It can also come from a better understanding of mathematics. Albert Einstein in his younger days complained "since the mathematicians have invaded the theory of relativity, I do not understand it myself" but 20 years later offered the opinion that "nature is the realization of the simplest possible mathematical ideas." But mathematicians rarely pursue art for art's sake. They are always out to discover and understand.

Here is Galileo again: "All truths are easy to understand once they are discovered; the point is to discover them." And for mathematicians knowing what is true, or discovering what is true, is a matter of analogy and metaphor, comparisons with other parts of mathematics –or more frequently from outside mathematics, in the physical sciences. Mathematical proof is often another question involving technique, knowledge of what others have done and sheer invention. Nor is mathematics a static discipline. It has its own internal dynamics: some fields develop and brush against neighbouring areas, some settle down to steady progress for a few decades and then explode. Some of the growth areas of the 1960s, for example, when resources were poured into science, became quiescent twenty years later but then sprang back into life. Or to take a longer-term view one might pick out Bernhard Riemann's work in the mid 19th century on differential geometry; its subsequent development in higher dimensions by Gregorio Ricci-Curbastro in 1904 prepared it for its phenomenal expansion when it was seen as the language in which to express Einstein's general relativity.

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