TOPOLOGICAL GAMES TO SOLVE PROBLEMS OF REAL LIVES

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Abstract : This paper is concerned with the study of topological approach to Game theory. By defining an ideal I over a topological space X an infinite positional game is played by two players pursuer and evader. This game is applied in covering properties of some topological spaces. Here topological spaces are assumed to be Hausdorff.

Key words: topological spaces, C scattered, paracompact space, closure preserving covers, topological product.

1. INTRODUCTION

Covering properties of topological spaces plays very important role in topology. A. Lelek (1969) has studied some covering properties of spaces.

'C scattered and paracompact space' and 'closure preserving covers' are studied by R. Telegarsky (1971). In this section we apply the game $G(I, X)$ over an ideal of a topological space in covering properties of some topological spaces to obtain useful results.

Here topological spaces are assumed to be Hausdorff. Positive numbers are denoted by i, j, k, m, n etc. and μ denotes an infinite cardinal number. In this section the following classes of spaces are considered:

DC : The class of all spaces which can be decomposed into a discrete collection by compact sets.

DCμ : The class of all spaces which can be decomposed into a discrete collection by μ-compact sets.

Definition 1. A space is said to be μ-compact if each of its open cover of power μ has a finite subcover.

A subset $A \times B$ of a topological product space $X \times Y$ is called a rectangle. A rectangle E is said to be :

(a) Co-zero if E^1 & E^{11} are co-zero in $X \times Y$;

(b) Zero if E^1 & E^{11} are zero in $X \times Y$

(c) Open if E^1 & E^{11} are open in $X \times Y$

(d) Closed if E^1 & E^{11} are closed in $X \times Y$;

where E^1 & E^{11} are the projections of E into X and Y respectively so that $E = E^1 \times E^{11}$.

2. STRONGLY RECTANGULAR PRODUCTS

A topological product $X \times Y$ is said to be strongly rectangular if each locally finite open cover of $X \times Y$ has a locally finite refinement by co-zero rectangles. From above definitions the following conditions are seen to be equivalent:

(a) The product $X \times Y$ is strongly rectangular.

(b) Each finite open cover of $X \times Y$ has a locally finite refinement by co-zero rectangles.

(c) For each closed subset F and each open set U of $X \times Y$ with $F \subset U$, there is a locally finite collection W by cozero rectangles s.t.
$F \subset \cup W \subset U.$

(d) $X \times Y$ is normal and for each zero-set F and each cozero-set U of $X \times Y$ with $F \subset U$, there is a locally finite collection W by cozero rectangles such that
$F \subset \cup W \subset U.$

(e) There exists a continuous map
$f : X \times Y \to [0, 1]$
such that
$$f(x, y) = \sum_{t \in T} g_t(x) h_t(y),$$
where $g_t : X[0, 1]$ and $h_t : Y \to [0, 1]$ are continuous.

Definition 2. A space is said to μ-paracompact if each of its open cover of power ($\leq \mu$) has a locally finite open refinement.

We state the following two lemmas of K. Morita and T. Hoshina.

Lemma 3. Let F be a μ-compact closed subset of a normal space X and let Y be a completely regular space with $\chi(Y) \leq \mu$ and $y \in Y$. If O be a finite collection of open sets in $X \times Y$ s.t. O Covers $F \times (Y)$, then there exists a finite collection

$L = \{U_i \times V/i = 1, \ldots, k\}$

by cozero rectangles in $X \times Y$ s.t.

$F \subset \cup \{ U_i/i \leq k\}, y \in V$ and L refines O.

Here x be cardinal function denoting character.

Lemma 4. Let $\{Z_\alpha/\alpha \in \Omega\}$ be a locally finite collection by zero set in X s.t. there exists a locally finite collection $\{H_\alpha/\alpha \in \Omega\}$ by cozero-sets in X with $Z_\alpha \subset H_\alpha$ for all $\alpha \in \Omega$. Then $\cup\{Z_\alpha/\alpha \in \Omega\}$ is a zero set in X.

Now we establish the following important theorem.

Theorem 5. Let X be a collection wise normal space and player P has a winning strategy in G (DC_μ, X) and let Y be a paracompact space $\chi(Y) \leq \mu$. Then $X \times Y$ is collection wise normal.

Proof. Let a winning strategy of player P in G (DC_μ, X) be s and O be any open cover of $X \times Y$ such that

(A) O contains a finite subcollection O which covers F for each μ-compact closed subset F in $X \times Y$.

We construct the following:

(i) A sequence $\{L_n/n \geq 0\}$ of collections by cozero rectangles,

(ii) An inverse system $\{< T_n, \psi_m^n > / n \geq 0, m \leq n\}$ of a sequence of collection T_n, by zero retangles,

(iii) A map ψ_m^n s.t. $\psi_m^n : T_n \to T_m$ where $n \geq 0$, $m \leq n$,

(iv) A sequence $\{S_n/n \geq 0\}$ of collections by cozero rectangles, and

(v) A countable collection $\{V(n, m)/n, m \geq 0\}$ of cozero sets in $X \times Y$, where $L_0 = \{\phi\}$, $T_0 = S_0 = \{X \times Y\}$, and $V(0, m) = X \times Y$ for all $m \geq 0$, satisfying the following conditions for all $n \geq 1$;

(1) L_n is locally finite in $X \times Y$.

(2) T_n is locally finite in $X \times Y$.

(3) Each $U \times V$ L_n is contained in some G_0.

(4) $Z \subset \psi_{n-1}^n(Z)$ for all $Z \in T_n$.

(5) If $Z_n \in Z_{n-1} \in T_{n-1}$ and then there exist some $Z_n \in T_n$ s.t. $P \in Z_n$ and $\psi_{n-1}^n(Z_n) = Z_{n-1}$.

(6) If $\{Z_0, Z_1, ..., Z_n\} \in \underset{i=1}{\overset{n}{\pi}} T_i$ satisfies $i_j(Z_i) = Z_j$ for all $1 \leq i \leq n$, then the finite sequence $(E_0, E_1, \ldots, E_{2n})$ defined by $E_0 = X$, $E_{2i} = Z_i^1$ and $E_{2i-1} = s(E_0, E_1, \ldots, E_{2i-2})$ for all $1 \leq i \leq n$ is admissible for $G(DC_\mu, X)$.

(7) $S_n = \{H(Z)/Z \in T_n\}$ S.t. $Z \subset H(Z)$ for all $Z \in T_n$.

(8) S_n is locally finite in $X \times Y$.

(9) $\cup T_n = \cap\{V(n, m)/m \geq 0)$, where $V(n, 0) = X \times Y$.

(10) Cl $V(n, m) \subset V(n, m\text{-}1) \cap V(n\text{-}1, m)$, for all $m \geq 1$.

(11) $\cup L_n \subset V(n\text{-}1, n\text{-}1)$.

Let $\{L_i / i \leq n\}$,

$\{[T_i, \psi_m^i]/ m \leq i \leq n\}$,

$\{S_i, i \leq n\}$,

and $\{V(i, m)/i \leq n, m \geq 0\}$,

satisfying (1) to (11) for all $i \leq n$, have been already constructed.

Now we fix $Z \in T_n$.

By (6), we define

$E_0 = X$;

$E_{2i} = \psi_i^n(Z)^1$;

$E_{2i\text{-}1} = s(E_0, E_1, \ldots, E_{2i\text{-}1})$

for all $1 \leq i \leq n$ and the finite sequence $(E_0, E_1, \ldots, E_{2n})$ is admissible for $G(DC_\mu, X)$.

Thus $s(E_0, E_1, \ldots, E_{2n})$ is the union of a discrete collection $\{F_\alpha/\alpha \in \Omega(Z)\}$ by α-compact closed subsets of Z^1.

Now we choose two discrete sets

$\{M_\alpha/\alpha \in \Omega(Z)\}$

and $\{C_\alpha/\alpha \in \Omega(Z)\}$

s.t. M_α is a cozero set in X. C_α is a zero set in X and $F_\alpha \subset M_\alpha \subset C_\alpha$ for all $\alpha \in \Omega(Z)$.

On applying lemma 3 and lemma 4, it follows from (A), (7), (9) and paracompactness of Y that a locally finite collection

$L_\alpha(Z) = \{U_{\lambda,i} \times V_{\lambda,i}/i = 1,2, \ldots, k_\lambda \text{ and } \lambda \in \wedge(\alpha)\}$

by cozero rectangles exists for all $\alpha \in \Omega(Z)$, satisfying the following:

(i) $F \subset U_\lambda = \cup\{U_{\lambda,i}/i \leq k_\lambda\} \subset M_\alpha$ for all $\lambda \in \wedge(\alpha)$.

(ii) $\{V_\lambda/X \in \wedge (\alpha)]$ is locally finite collection by cozero sets in Y, covering Z^{11}.

(iii) Each $U_{\lambda,i} \times V_\lambda$ is contained in some $G \in 0$.

(iv) $U_\lambda \subset H(Z)^1$ and $V_\lambda \subset H(Z)^{11}$ for all $\lambda \in \wedge(\alpha)$.

(v) $U_\lambda \times V_\lambda \subset V(n, \text{n})$ for all $\lambda \in \wedge(\alpha)$.

If we put $L_{n+1} = \cup\{L_\alpha(Z)/Z \in T_n$ and $\alpha \in \Omega(Z)\}$ and $L = \cup \{L_n/n \geq 0\}$, then clearly, L covers $X \times Y$ with the following:

Claim (i) $\cap\{Z_n/n \geq 0\}$ = for all $\{Z_n\} \in \overset{\lim}{\longleftarrow}\{T_n, \psi_m^n\}$.

$\therefore$ L is a σ-locally finite refinement of O by cozero rectangles by conditions (1) and (3).

Also, with the claim (2)

$\cap\{\cup T_n/n \geq 0\}$

$\Rightarrow L$ is locally finite in $X \times Y$.

Let $p \in X \times Y$. By claim (2), some $n_1 > 0$ can be taken s.t. $p \notin \cup T_n$.

By (9) and (10), some $n_0 \geq n_1$ can be taken s.t. $p \notin$ Cl $V(n_1, n_0)$.

Therefore, $p \in$ Cl $V(n_0, n_0)$. Also by (10) and (11), $P \notin$ Cl $(\cup\{\cup L_n/n \geq n_0\})$.

L is locally finite at p since $\cup \{L_n/n \geq n_0\}$ is locally finite in $X \times Y$.

Therefore the following result can be obtained: Each open cover O of $X \times Y$ satisfying the condition (A) has a locally finite refinement L by cozero rectangles. Put $O = \{X \times Y/\cup\{DC\}\}$ such that $c \neq \eta \in \Omega$.

Since each countably compact closed subset in $X \times Y$ intersects at most finite many elements of $\{DC\}$, O satisfies the condition (A).

It follows from the above result that O is normal cover of $X \times Y$. Clearly, there exists a disjoint collection of open sets separating $\{DC\}$ in $X \times Y$.

Hence $X \times Y$ is collection wise normal.

Corollary 6. Let X be a collection wise normal space and player P has a winning strategy in $G(DC_\mu, X)$ and let Y be a paracompact space $X(Y) \leq \mu$. Then $\dim X \times Y \leq \dim X + \dim Y$.

Proof. Pasynkov, B.A. [63] has established that dim $X \times Y \leq$ dim X+ dim Y holds if $X \times Y$ is rectangular.

A strongly rectangular product $X \times Y$ is clearly rectangular. From this result and above theorem 5, this corollary can easily be obtained.

Theorem 7. Let X be a collection wise normal space which a σ-closure preserving closed cover by μ-compact sets and Y be a paracornpact space with $x(Y) \leq \mu$. Then $X \times Y$ is collection wise normal, paracompac and strongly rectangular.

Proof. Let the topological space X has a σ-closure preserving closed cover by μ-compact sets.

Let $X = \cup\{X_n\}$; where $\{X_n\} \in P(X)$; and $P(I, X_n) \neq \phi$ for each $n \in N$. Suppose, $s_0 \in P(I, X_0)$, and $\{k_n\}$ be the set N_0 ordered as a strictly increasing sequence such that $k \in N$.

Let $\{E_n\}$ be a play of $G(I, X)$, then

$(X_n \cap E_{2k0}, X_0 \cap E_{2k0+1}, X_0 \cap E_{2k1}, X_0 \cap E_{2k1+1}, \ldots)$

is a play of $G(I, X_0 \cap E_{2k0})$.

Thus by theorem 5, $P(I, X_0 \cap E_{2k0}) \neq \phi$.

Now let

$$
\begin{aligned}
& S_0^1 \in P(I, X_0 \cap E_{2k0}) \\
& \Rightarrow \cap(X_0 \cap E_{2kn}) = \phi \\
& \Rightarrow X_0 \cap \cap\{E_{2kn}\} = \phi \\
& \Rightarrow \cap\{E_{2kn}\} = \cap\{E2_n\} \\
& \Rightarrow X_0 \cap \cap\{E_{2n}\} = \phi \\
& \therefore P(I, X) = \phi.
\end{aligned}
$$

It can be also proved that player P has winning strategy in $G(DC_\mu, X)$.

Let Y be a paracompact space with $\chi(Y) \leq \mu$. Then by theorem 5, $X \times Y$ is collectionwise normal, μ-paracompact and strongly rectangular.

Hence with the help of theorem 7 we have the following results:

(i) If X is a collectionwise normal space which has a σ-closure preserving closed cover by countably compact sets and Y is a paracompact, then the product space $X \times Y$ is collectionwie normal, countably paracompact and the above inequality dim $X \times Y \leq$ dim X + dim Y holds.

(ii) If X is a paracompact space which has a σ-closure preserving cover by compact sets and Y is a paracompact space, then the product space $X \times Y$ is paracompact and the above inequality dim $X \times Y \leq$ dim X + dim Y also holds in this case.

Conclusion

This paper a very important result is proved that is if X be a collection wise normal space and player (pursuer) has a winning strategy in a game over DC_μ class and Y be a pazacompact space with $X(Y) \leq \mu$ then the product space $X \times Y$ is collection wise normal. It is also shown that if the above X be with a σ-closure preserving closed cover by μ-compacts and Y be same as above then product $X \times Y$ is collection wise normal, μ-paracompact and strongly rectangular.

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