Study and some Result on Non expansive Mapping in linear 2 normed spaces.

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INTRODUCTION— The notion of linear 2- normed spaces was introduced by S. Gahler. He further studies the topological studies of 2-normed spaces. Iseki introduced the notion of non-expansive mapping in 2- normed spaces. Then mathematician like Diminni and white further studied non-expansive mapping in linear 2- normed spaces and obtained the results of Iseki as their corollaries and they contributed a lot for the extension of this branch of mathematics, physics and other Science.

KEYWORD — 2- normed spaces, non-expensive mapping, convex subset

1. Let X be a linear space of dimension greater than 1 and let $||\cdot,\cdot||$ be a real valued function defined on $X \times X$ such that :
   1. $|| a, b|| = 0$ if any only if and b are linearly dependent,
   2. $|| a, b|| = ||b, a||$,
   3. $||a, \alpha b|| = | \alpha | ||a, b||$, were α is real,
   4. $||a + b, c|| < ||a, b|| + ||a, c||$.

   $||\cdot,\cdot||$ is called a 2-norm on X and (X, $||\cdot,\cdot||$is a linear 2-normed space. By condition 2 and 4, a 2-norm is non-negative.

Definition **:** If K is a convex subset of X, a mappings $T : K \rightarrow X$ is said to be non-expansive if for every $x, y \in K$ and $z \in X$,

1. $||T(x) – T (y), z|| \leq ||x – y, z||$.

   In the following, the real number system will be denoted by R. Also, a subset of L of x of the form $\{x_1 + ax_2 : a \in R\}$, where x_2 is non-zero, will be called a line. $\alpha \in$

Theorem **:** Let K be a convex set which contains a least 2 elements and is none a subset of line. Then, T is non-expansive if and only if there is a $c \in R$ and there is a point $z_0 \in X$ such that $|c| < 1$ and $T(x) = cx + z_0$, for every $x \in K$.

Proof- Since all functions of the above type are non-expansive, we need show only that all non-expansive maps are of this type.

1. Assume first the $0 \in K$ and $T (o) = 0$. Then, for every $x \in X$,
2. $||T (x), z||<|| X, Z||$.

   Therefore, for each $x \in K$, there is a real number $g(x)$ such that $T(x) = g(x)x$.

If x and y are independent elements of K, then $\frac{1}{2}$ (x + y) ∈ K also, and by (1),

$$\left(T\frac{x+y}{2}\right)--T(x), x--y\,|||<||\,\frac{x+y}{2}x-y\,||=0.$$

Therefore, there is a k ∈ R such that

$$\left(T\frac{x+y}{2}\right)--T(x)=k(x-y)$$

$$g\left(\frac{x+y}{2}\right)\left(\frac{x+y}{2}\right)-g(x)x=k(x-y).$$

Then,

$$\left[\frac{1}{2}g\left(\frac{x+y}{2}\right)-g(x)-k\right]x=-\left[k+\frac{1}{2}g\left(\frac{x+y}{2}\right)\right]y$$

which implies that g(x) $=g\left(\frac{x+y}{2}\right)$ by the independence of x and y. Since a similar argument shows that g(y)= $g\left(\frac{x+y}{2}\right)$, it follows g(x) = g(y) whenever x and y are independent.

If x and y are non-zero, independent elements of K, then since K is not a subset of a line, there is a z ∈ K such that z and x and z and y are independent. By the arguments used above, g(x) = g(z) = g(y).

Therefore, g(x) = g(y) for all non-zero x, y ∈ K. Since T(0) = 0, there is a real number c such that T (x) cx for every x ∈ K. Finally, (2) implies that |c| < 1.

2. For arbitrary T and K which satisfy the hypotheses, choose and x ∈ K' = {x – x_0 : x ∈ K}. Then K' is not contained in a since K is not a subset of a line, and x ∈ K'. Define S : K' → x by

||S(x – x_0) – S(y – x_0), z||=|| T (x) – T (y), z ||.

< ||x –y, z||

=||(x – x_0) – (y – x_0), z||.

Hence, S is non-expansive on K' and

S(0) = S(x –x_0) = T (x_0) – T (x_0) = 0

By part 1, there is a c ∈ R such that |C| < 1 and for every x ∈ K,

S(x – x_0) = c (x – x_0).

Therefore, for every x ∈ K,

T(x) = cx + T (x_0) – x_0.

The following example shows that the characterization fails if K is contained in a line.

Example: Suppose K is subset of the line L = T(x) = cx + T(x_0) – x_0.

Define T : K → X by $T(x_1+\alpha x_2)=(\sin\alpha)x_2$.

Then, if $x_1 + \alpha x_2$ and $x_1 + \gamma x_2$ are in K and z ∈ X,

$$\| T(x_1 + \alpha x_2) - T \| (x + \gamma x_2), z \| = \| | \sin\alpha - \sin\gamma | \| x_2, z \| < \alpha - \gamma | \| x_2, z \|$$

$$= \| (x_1 + \alpha x_2) - (x + \gamma x_2), z \| .$$

Hence, T is a non-expansive mapping which does not satisfy Theorem 1.

For convex sets which are subsets of lines, we have the following characterization of non-expansive mappings.

***Theorem*:** Suppose K is a convex subset of line L = $\{x_1 + \alpha x_2 : \alpha \in R\}$, where $x_1 \in$ K, and let $\{\alpha : x_1 + \alpha x_2 \in R\}$. Then, T : K $\rightarrow$ X is non-expansive if and only if there is a function g : A g(0) = 0 and T $(x_1 + \alpha x_2) = g(\alpha)\, x_2 + T\,(x_1)$.

***Proof*:** Again, since the sufficiency of the above conditions is clear, we deed only to prove the necessity.

1. Assume $X_1 = 0$ and T(0) = 0. Then, for every $\alpha \in$ A and z ∈ X, (3) $\|T\,(x_2), z\| < \|\alpha x_2, z\|$.

   Therefore, for every non-zero $\alpha \in$ A, there is a real number g (α) such that $|g(\alpha) - g(\gamma)| < |\alpha - \gamma|$ for every $\alpha, \gamma \in$ A.

2. If x_1 = 0 or $T(x_1)$ = 0 let $K' = \{\alpha x_2 : \alpha \in A\}$. Then, K' is convex, 0 ∈ K', and $K' = \{\alpha x_2 : \alpha \in R\}$. Define S : K' $\rightarrow$ X by

   $$S(\alpha x_2) = T(x_1 + \alpha x_1) - T(x_1)$$

   for every $\alpha \in$A. Note that S(0) = 0 and for $\alpha, \gamma \in$ A and z ∈ X,

   $$\| S(\alpha x_2) - - S(\gamma x_2), z \| = \| T(x_1 + \alpha x_2) - - T(x_1 + \gamma x_2), z \| < \| \alpha x_2 - \gamma x_2, x \| .$$

Therefore, since S and K' satisfy the assumptions made in part 1, it follows that there is a function g : A $\rightarrow$ R such $S(\alpha x_2) = g(\alpha) x_2$. Hence, for every $\alpha \in$ A, $T(x_1 + \alpha x_2) = g(\alpha) x_2 + T(x_1)$.

It is known that in a strictly convex 2-normed space, the set F(T) of fixed points of a non-expansive T is always a convex set. This result can now be proven for any 2-normed space.

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