Generalized Laguerre Transform and convolution of Laguerre Transformable Generalized Functions

P.N. PATHAK[1] , VARUN KUMAR[2] and RAGHAVENDRA SINGH[3]

[1]Deptt of Mathematics, AIHE, Axis Colleges Kanpur, [2]Deptt of Mathematics Sabarmati university Ahmadabad, Gujarat, [3]Deptt of Mathematics IITM Bilhaur Kanpur

Abstract:

In this paper we deal with Laguerre polynomials which are known to form a sequence of orthogonal polynomials in the interval (0,∞) with respect to the weight function $x^{\alpha}.e^{-x}$. The members of $L_2(I)$-space where I = (0, ∞) are expressible in series of these polynomials and the coefficients in such series come out to be as an integral transform which is known as Laguerre transform. Zemanian [13] has given the theory of extension of Laguerre transform to the space of generalized functions. A testing function space as a subspace of $L_2(I)$-space has been constructed and hence generalized Laguerre transforms of members of dual space of this space have been defined.

1. Introduction:

The various methods for expansion of certain Schwartz distributions (generalized functions) with respect to different orthonormal systems and that for generalization of the integral transforms to the certain class of generalized functions by using these expansions have been developed by several authors. As a by-product of these expansions certain inversion formulae for the integral transforms of generalized functions are obtained therefore each of the methods of such expansions arises a whole new class of generalized integral transformations. The resulting generalized integral transforms are the finite Fourier transform, the Laguerre transform, the Hermite transform, the Jacobi transform with its special case: the Legndre, Chebyshev and Gergenbauer transforms and finally the finite Hankel transforms.

In 1966, Zemanian [11] has given new approach of expanding of certain Schwartz distributions into series of orthonormal functions which are eigen function of the self adjoint differential operator and he developed a straight forward technique to generalized in a distributions way a variety of integers transforms. Specially his work based on L_2-convergence theory of orthonormal series and in particular, he also identified few spaces of distributions corresponding to certain orthonormal system. e.g. space S_n' of tempered distributions in the case of Hermite system.

Further Zemanian`s method has been extended to all regular C^∞, self-adjoint ordinary differential operators by Judge [3] and in the same year, Guillemot-Teissiers [8] developed a technique of expanding of tempered distributions in orthonormal series of Laguerre polynomials.

In 1978, Panday and Pathak [4] discussed the same kind work as in Zemanian [13] by extending the L_1 convergence theory of orthogonal series in the distributional sense. Further in 1983, following Zemanian, Pathak [6] extended his work by applying L^p-convergence theory. In between certain summability methods also were employed by Ditzion [1], Pathak[5].

Recently in 1990 Duran [2] discussed expansion of distributions in the orthonormal series of the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$. Specially he extended the result of Guillenot-Tessiers and obtained some new and interesting properties applying these results. In this special case Duran`s space of distribution coincides with space S_n' of tempered distribution.

2. System of Complete Orthonormal Functions ψ_n

In this section we describe the orthonormal system $\{\psi_n\}$ as below:

We know that the sequence of polynomials $L_n^{(\alpha)}(x)$,n=0,1,2 and Re(α) >-1 forms an orthogonal set over the interval (0,∞) with respect to weight function $x^\alpha e^{-x}$, i.e.

$$\int_0^\infty x^\alpha e^{-x} L_n{}^{(\alpha)}(x) L_m{}^{(\alpha)}(x)dx = 0 \quad if \quad m \neq n \tag{2.1}$$

We also have

$$\int_0^\infty x^\alpha e^{-x} (L_n{}^{(\alpha)}(x))^2 dx = \frac{\Gamma(1+\alpha+n)}{n!} \tag{2.2}$$

Moreover, from above we can show that the functions

$$\psi_n(x) = \left(\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} x^{\alpha/2} e^{-x/2} L_n{}^{(\alpha)}(x) \tag{2.3}$$

n=0, 1, 2,…, Re(α)> -1, form a system of complete orthonormal functions in the Hilbert space $L_2(I)$ where I= (0, ∞) and these functions $\psi_n(x)$ are eigen functions of the operator.

$$\eta = x^{-\alpha/2} e^{x/2} D x^{\alpha+1} e^{-x} D x^{-\alpha/2} e^{x/2} \tag{2.4}$$

corresponding eigen values $\lambda_n = -n$

In particular, for α = 0, the Laguerre system is given as

I=(0,∞)

$$\eta = e^{x/2} \frac{d}{dx} x e^{-x} D e^{x/2} = xD^2 + D - \frac{x}{4} + \frac{1}{2} \tag{2.5}$$

And the complete system of eigen functions for η consists of the Laguerre functions

$$\psi_n(x) = e^{-x/2} L_n(x) \quad , n = 0,1,2,\ldots \tag{2.6}$$

where $L_n(x) \quad = \quad \sum_{m=0}^{n} \binom{n}{m} \frac{(-x)^m}{m!}$ (2.7)

and the corresponding eigen values are $\lambda_n \; = \; -n$

3. Testing Function space $A\left(L_n^{(\alpha)}\right)$

In order to define generalized Laguerre transform, we follow Zemanian [13] and so we now construct a subset of $L_2(I)$ which serves as a testing function space for the generalized functions whose generalized Laguerre transform will be defined. Let us denote this subset of $L_2(I)$ by $A(L_n^{(\alpha)})$.

Let $A(L_n^{(\alpha)})$consists of all functions φ (x) that possess the following three properties:

(i) φ (x) is defined, real valued and smooth on $I=(0, \infty)$

(ii) For each k=0, 1, 2, ……

$$\gamma_k(\varphi) \; = \; \gamma_0(\eta^k \varphi) \; = \; \left[\int_0^\infty |\eta^k \varphi(x)|^2\right]^{1/2} < \; \infty \qquad (3.1)$$

where operator η is defined by (2.4), and

(iii) for each n and k as above, we have

$$(\eta^k \varphi, \psi_n) \quad = \quad (\varphi, \eta^k \psi_n) \qquad (3.2)$$

where $\qquad \psi_n(x) = \left(\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} x^{\alpha/2} e^{-x/2} L_n{}^{(\alpha)}(x)$

It can easily be proved that $A\left(L_n^{(\alpha)}\right)$ is a vector space this vector space is made into a topological vector space by defining a topology generated by separating collection of semi norms γ_k, k=0,1,2,……defined by (3.1). This topology having a countable local base is metrizable through the metric defined by

$$d(\varphi, \psi) \; = \; \sum_{k=0}^{\infty} \frac{2^{-k} \gamma_k(\varphi-\psi)}{1+\gamma_k(\varphi-\psi)} \qquad (3.3)$$

where $\varphi,\psi \in A(L_n^{(\alpha)})$. Following Rudin [7] and Zemanian [10], it is clear that d is complete hence $A(L_n^{(\alpha)})$ is a Frechet space and therefore $A(L_n^{(\alpha)})$ turns out to be a testing function space.

We define the Laguerre transform of $\phi \in A(L_n^{(\alpha)})$ by the relation

$$T(\varphi)=\Phi(n) = \int_0^\infty \varphi(x)\, \overline{\psi}_n(x) dx = (\varphi, \psi_n) \qquad (3.4)$$

where functions $\psi_n(x)$ are normalized Laguerre functions given by (2.3).

4. Space of Laguerre Transformable Generalized Function $A(L_n^{(\alpha)})$

We denote the dual of $A(L_n^{(\alpha)})$ by $A'(L_n^{(\alpha)})$ which is a space of generalized functions on which Laguerre transform will be defined. $A'(L_n^{(\alpha)})$ is also complete since $A(L_n^{(\alpha)})$ is a complete countable multinorm space.

We now discuss below some useful and important properties of the generalized function space $A'(L_n^{(\alpha)})$.

We define a differential operator η′ on $A'(L_n^{(\alpha)})$ where η′ is obtained by reversing the order in which differentiation and multiplication by functions $x^{\alpha/2}e^{-x/2}$ and $x^{\alpha+1}e^{-x}$ occur in $\eta = x^{\alpha/2}e^{-x}Dx^{\alpha+1}e^{-x}Dx^{\alpha/2}e^{x/2}$and replacing each D by –D. Then we see that η′=η. Thus the differential operator on $A'(L_n^{(\alpha)})$ is defined by the relation

$$(\eta f, \varphi) = (f, \eta\varphi), \quad f \in A'(L_n{}^{(\alpha)}), \quad \varphi \in A(L_n{}^{(\alpha)}) \tag{4.1}$$

Since η is a continuous linear mapping of A $(L_n^{(\alpha)})$ into A $(L_n^{(\alpha)})$ therefore it is also a continuous linear mapping of $A'(L_n^{(\alpha)})$ into $A'(L_n^{(\alpha)})$.

It is obvious from definition of $A'(L_n^{(\alpha)})$ that $D(I) \subset A(L_n{}^{(\alpha)})$ and convergence in D(I) implies convergence in $A(L_n^{(\alpha)})$. Consequently the restriction of any $f \in A'(L_n{}^{(\alpha)})$ to D(I) is a member of D′(I).

Since $A(L_n^{(\alpha)}) \subset L_2(I)$ and the dual of $L_2(I)$ is $L_2(I)$. Therefore $L_2(I) \subset A'(L_n{}^{(\alpha)})$

(iv) ∈′(I) is a subspace of $A'(L_n^{(\alpha)})$ where ∈′(I) is the space of all distributions whose support are compact subsets of I.

(v) For each $f \in A'(L_n{}^{(\alpha)})$ there exist a non negative integer r and a positive constant C such that

$$|(f, \varphi)| \leq C \max_{0\leq k\leq r} \gamma_k(\varphi) \tag{4.2}$$

for every $\varphi \in A(L_n{}^{(\alpha)})$, here r and C depend on f but not on φ.

Following Zemanian [13], we have described the expansions of members of $A'(L_n^{(\alpha)})$ with respect to this orthonormal system

If $f \in A'(L_n{}^{(\alpha)})$ then

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n \tag{4.3}$$

where the series converges is $A'(L_n^{(\alpha)})$. It is a fundamental theorem of our context.

Now by using these expansions, we define generalized Laguerre transform of member of $A'(L_n^{(\alpha)})$ and also establish its inversion formula. We also define the convolution of two Laguerre transformable generalized functions for α=o.

5. Laguerre Transform of Generalized Functions :

We now define the *generalized Laguerre transform* T of $f \in \boldsymbol{A'(L_n{}^{(\alpha)})}$ by means of the relation

$$T\{f\} = F(n) = (f, \psi_n) \tag{5.1}$$

where $\psi_n(x)$, n=0,1,2,… are normalized Laguerre function defined by (2.3)

The relation (4.3) defines inverse generalized Laguerre transform and it may be also written as

$$T^{-1}\{F(n)\} = f = \sum_{m=0}^{\infty} F(n)\,\psi_n \tag{5.2}$$

In particular, for α = 0 i.e. corresponding Laguerre system defined by (2.7), we denote our testing function space and space of generalized functions by $A(L_0)$ and $A^{'}(L_0)$ respectively.

Now in this case the generalized Laguerre transform of $f \in A^{'}(L_0)$ is defined by the relation

$$T\{f\} = F(n) = (f, L_n(x)e^{-x/2}) \tag{5.3}$$

And we have, If $f \in A'(L_0)$ then

$$f = \sum \quad (f, L_n(x)e^{-x/2})L_n(x)e^{-x/2} \tag{5.4}$$

which is defined inverse transform.

Further suppose $f \in A^{'}(L_0)$ is a distribution whose support is compact subset of I = (0,∞) then it can easily be shown that the generalized Laguerre transform T, defined by (5.3) establish the following operation transform formulas:

$$T[e^{-x/2}D\ e^{x/2}f(x)] = \sum_{k=0}^{n} F(k)$$

$$T[e^{-x/2}xD\ e^{x/2}f(x)] = nF(n) - (n+1)F(n+1)$$

$$T[e^{-x/2}D\ xe^{-x/2}f(x) + xf(x)] = (n+1)[F(n) - F(n+1)]$$

$$T[e^{-x/2}D\ xDe^{x/2}f(x)] = -(n+1)F(n+1)$$

$$T[e^{-3x/2}D\ xe^{x}De^{x/2}f(x)] = -2(n+1)F(n+1) + nF(n)$$

$$T[xf(x)] = -(n+1)F(n+1) + (2n+1)F(n) - nF(n-1)$$

These results are given by Zemanian [10].

"It is most important to mention there that the space A′(L_0) identifies with the space of tempered distributions of positive support **(**see Guillemot-Teissiers [8]**)"**

6. Convolution of Laguerre Transformable Generalized Functions

The convolution product *f*∗g of two generalized functions *f* and g in A′(L_0) is defined by the relation

$$\langle f{*}g, \varphi \rangle = \langle f(x), \langle g(y), \varphi(x+y) \rangle\rangle \tag{6.1}$$

for every $\varphi \in A(L_0)$. If we write $\psi(x) = \langle g(y), \varphi(x+y) \rangle$

Then following Zemanian [10], it can be easily proved that ψ ∈ A′(L_0) when φ ∈ A(L_0). Finally it can be shown that *f*∗g as defined by (6.1) is also a member of A′(L_0).

We now defined generalized Laguerre transform T of the convolution of *f* and g in A′(L_0) as follows.

From (5.3) we have

$T\{f * g\} = (f * g, L_n(t).e^{-t/2})$

$$= (f(x) * g(y), L_n(x+y)e^{-(x+y)/2}) \quad (6.2)$$

In the view of the following results

$$L_n(t) = L_n{}'(t) - L_n{}' - 1(t)$$

$$and \qquad L'_n(x+y) = \sum_{k=0}^{n} L_{n-k}(x) L_k(y)$$

The relation (6.2) becomes

$$T\{f * g\} = \sum_{k=0}^{n} (f(x),\ L_{n-k}(x)\ e^{-x/2})(g(y),\ L_k(y)\ e^{-y/2}) - \sum_{k=0}^{n-1} (f(x),\ L_{n-k-1}(x)\ e^{-x/2})(g(y),\ L_k(y)\ e^{-y/2}) \quad (6.3)$$

further if we assume that

$$f = \sum_{n=0}^{\infty} a_n\ L_n(t)\ e^{-t/2} \text{ and } g = \sum_{n=0}^{\infty} b_n\ L_n(t)\ e^{-t/2}$$

where $a_n = (f, L_n(t)\ e^{-t/2})$ and $b_n = (g, L_n(t)\ e^{-t/2})$

then the equation (6.3) reduces to

$$T\{f * g\} = \sum_{k=0}^{n} [b_k(a_{n-k} - a_{n-k-1})] \quad (6.4)$$

In future we will also define convolution of two Leguerre transformable generalized functions in the general case i.e. for generalized functions in $A'(L_n^{(\alpha)})$.

7. We now illustrate some useful examples of the generalized Laguerre transforms of some $f \in A'(L_0)$ and also discuss their Fourier Laguerre expansions.

(I) Let $f = \delta$, delta functional then

$$T\{\delta\} = (\delta, L_n\ e^{-t/2}) = \int_0^{\infty} \delta(t) L_n(t) e^{-t/2} dt$$

$$= L_n(0) = 1$$

and then its Fourier Laguerre expansion is

$$\delta = \sum_{n=0}^{\infty} L_n(t) e^{-t/2}$$

(II) $T\{D^k\delta\} = (D^k\delta\ \ L_n(t)e^{-t/2})$

$$= (\delta, D^k L_n(t) e^{-t/2})$$

$$= \left(\delta, \sum_{m=0}^{k} \binom{k}{m} D^m L_n(t) D^{k-m}(e^{-t/2})\right) \quad by\ Leibritz\ Theorem$$

$$= \left(\delta, \sum_{m=0}^{k} \binom{k}{m} \left(\frac{-1}{2}\right)^{k-m} (-1)^m L_{n-m}{}^m(t) e^{-t/2}\right)$$

$$= \left(\sum_{m=0}^{k} \left(\frac{1}{2}\right)^{k-m} \binom{k}{m} \binom{n}{m} \right)$$

And therefore its Fourier Laguerre expansion is

$$D^k \delta = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{k} \left(\frac{1}{2}\right)^{k-m} \binom{k}{m} \binom{n}{m} \right] L_n \ (t) e^{-t/2}$$

(III) $T\{\delta(t-a)\} = (\delta(t-a), L_n(t)e^{-t/2})$

$$= \int_0^\infty \delta(t-a) L_n(t) e^{-t/2}\, dt$$

$$= L_n(a) e^{-a/2} \qquad if\ a \geq 0$$

∴ Its Fourier Laguerre expansion is

$$\delta(t-a) = \sum_{n=0}^{\infty} L_n(a) e^{-a/2} L_n(t) e^{-t/2}$$

(IV) Let $f(t) = \frac{t\ e^{t/2}}{e^{at-1}}$ Then

$$= \left(\frac{t\ e^{t/2}}{e^{at-1}}, L_n(t) e^{i/2} dt \right)$$

$$= \int_0^\infty \frac{t\ e^{t/2}}{e^{at-1}}, L_n(t) e^{-i/2} dt$$

$$= \int_0^\infty \frac{t}{e^{at}-1}, L_n(t) dt$$

$$= \sum_{k=0}^{N} \frac{(-1)^k}{k!} \binom{n}{k} \int_0^\infty \frac{t\ e^{t/2}}{e^{at}-1} dt$$

$$= \sum_{k=0}^{N} \frac{(-1)^k}{k!} \binom{n}{k} \frac{1}{\alpha^{k+2}} \Gamma(k+2) \zeta(k+2)$$

Therefore

$$T\left\{ \frac{t\ e^{t/2}}{e^{at}-1} \right\} = \sum_{k=0}^{n} (-1)^k \frac{(k+1)}{a^{k+2}} \binom{n}{k} \ \zeta(k+2)$$

where $Re(a) \geq \frac{1}{2}$ $and\ \zeta$ is the zeta function of Reimann and hence

$$\frac{t\ e^{t/2}}{e^{at-1}} = \sum_{k=0}^{n} \left(\sum_{k=0}^{n} (-1)^k \frac{(k+1)}{a^{k+2}} \binom{n}{k} \ \zeta(k+2) \right) L_n(t) e^{-t/2}$$

(V)

$$T(\chi_{[0,\infty)}(t)) = (\chi_{[0,\infty)}(t), L_n(t) e^{-t/2})$$

$$= \int_0^\infty L_n(t) e^{-t/2} dt = (-1)^n\, 2$$

And then its expansion is

$$\chi_{[0,\infty)}(t) = 2 \sum_{n=0}^{\infty} (-1)^n L_n(t) e^{-t/2}$$

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