NON - COMMUTATIVE REPRESENTATION OF SEMI GROUP MEASURES

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ABSTRACT

In this paper we extend some properties of probability measures, in its non-commutative frame work under the assumption of convolution semigroups and the associated semigoup of operators on the L^2 space of normalised Haar measure, when G is the real line. Using Peter-Weyl theory, we extend some properties of pseudo differential operators on compact groups. We show that the Hunt semigroup and its generator are pseudo differential operators in the sense of Ruzhansky and Turunen. We show here that the generator has the same Sobolev regularity as the Laplacian. We derive the transition kernel for convolution semigroups of cenetral measures. We derive a necessary and sufficient condition for the semigroup to be trace-class for any positive time which is equivalent to that the corresponding probability measure has a squareintegrable density.

KEY WORDS

Convolution semigroups, Central measure, Symmetric measure, Gaussian measure, Fourier transform, Brownian motion.

1. INTRODUCTION

Let G be a compact group with neutral element e and let M(G) be the set of all probability measures defined on (G, B(G)) where B(G) is the Borel σ −algebra of G. $\mu \in$ M(G) is central or conjugate- invariant if $\mu(\sigma A\sigma^{-1}) = \mu(A)$ for all $\sigma \in G, A \in B(G)$ It is said to be symmetric if $\mu(A^{-1}) = \mu(A)$ for all $A \in B(G)$. Let $M_C(G)$ $(M_S(G))$ be the subsets of $M(G)$ comprising central (symmetric) measures and let us define $M_{C,S}(G) = M_C(G) \cap M_S(G)$. Normalised Haar measure on G is denoted d_π when integrating functions of $\sigma \in G$.

Let $\hat{G}$ be the set of all equivalence classes of irreducible representations of G. We identify equivalence classes with a particular representative element. The trivial representation as denoted by δ . Each $\pi \in \hat{G}$ acts as a $d_\pi \times d_\pi$ unitary matrix on a complex linear space V_π having dimension d_π . We define the Fourier transform of each $\mu \in M(G)$ to be the Bochner integral

$$\hat{\mu}(\pi) = \int_C \pi(\sigma)\mu(d\sigma). \quad \text{------------------} \quad 1 \quad \text{------------------------(1)}$$

Where $\pi \in \hat{G}$. We use the relation

$$\widehat{\mu * v}(\pi) = \hat{\mu}(\pi)\hat{v}(\pi).$$

INFINITELY DIVISIBLE PROPERTIES OF CONVOLUTION SEMIGROUP OF PROBABILITY MEASURES

2. THEOREM

If G is a compact Lie group and $\mu \in M_{C,S}(G)$ is infinitely divisible then for each $\pi \in \hat{G}$ there exists $\alpha_\pi \leq 0$ such that $\hat{\mu}(\pi) = e^{\alpha\pi}\mathrm{I}\pi$.

Proof

Heyer derived that μ may be embedded as μ_ι into a vaguely continuous convolution semigroup of probability measures $(\mu_t, t \geq 0)$ where μ_0 is normalised Haar measure on a closed subgroup of G. We thus find that for each $\pi \in \hat{G}, (\hat{\mu}_t(\pi), t \geq 0)$ is a strongly continuous contraction semigroup of matrices acting on V_π we write $\widehat{\mu_t}(\pi) = \hat{\mu}_0(\pi)e^{tA\pi}$ for all $t \geq 0$ where A_π is a $d_\pi \times d_\pi$ matrix. Since $\mu_1 \in M_{c,s}(G)$ there exists $\lambda_\pi \in R$ such that

$$\hat{\mu}_1(\pi) = \hat{\mu}_0(\pi)e^{A\pi} = \lambda_\pi \mathrm{I}_\pi \ldots\ldots (*) \text{ ------------------------------------(2)}$$

If $\lambda_\pi = 0$, (2) with $\alpha_\pi = -\infty$ we assume that $\lambda_\pi \neq 0$. Since $\mu_1 = \mu_1 * \mu_0$ we obtain the relation

$$\hat{\mu}_0(\pi)e^{A\pi}\hat{\mu}_0(\pi) = \lambda_\pi \mathrm{I}_\pi$$

Also, multiplying both sides of (2) by $\hat{\mu}_0(\pi)$ we get

$$\hat{\mu}_0(\pi)e^{A\pi}\hat{\mu}_0(\pi) = \lambda_\pi\hat{\mu}_0(\pi) \text{ --(3)}$$

It thus follows that $\hat{\mu}_0(\pi) = \mathrm{I}_\pi$ and hence $H = \{e\}$. Also $A_\pi = \alpha_\pi \mathrm{I}_\pi$ where $\alpha_\pi \in R$ and $\lambda_\pi = e^{\alpha\pi}$. But $\hat{\mu}_1(\pi)$ is a contraction on V_π and hence $\alpha_\pi \leq 0$. Hence the theorem is proved.

3. THEOREM

(i) The measure $\mu_{\lambda\gamma}$, is central if and only if γ is Fourier transform

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(ii) The measure $\mu_{\lambda\gamma}$, is symmetric if and only if γ is Fourier transform

Proof

Part (i) is trivial and derivable from the result due to S. Saud. Also, conversely if $\mu_{\lambda\gamma}$, is central then for all $g \in G, \pi \in \hat{G}$,

$$\pi(g)\,\widehat{\mu_{\lambda\gamma}}(\pi)\pi(g^{-1}) = \widehat{\mu_{\lambda\gamma}}(\,\widehat{\mu_{\lambda\gamma}}) \text{ ----------------------------------(4)}$$

$$\widehat{\mu_{\lambda\gamma}}(\pi) = \exp\{\lambda(\pi(g)\hat{\gamma}(\pi)\pi(g^{-1}) - \mathrm{I}_\pi)\}$$

By an appropriate application of unifqueness of Fourier transforms and injectivity of the exponential map o matrices, we obtain the relation

$$\pi(g)\,\hat{\gamma}(\pi)\,\pi(g^{-1}) = \hat{\gamma}(\pi) \text{ ---(5)}$$

for all $g \in G, \pi \in \hat{G}$. Hence the theorem is proved.

We thus find that a probability measure is symmetric if and only if its Fourier transform comprises self-adjoint matrices. We conclude that a central probability measure μ is a compound Poisson distribution if and in only if there exists $\lambda > 0$ and a central probability measure γ with $\hat{\gamma}(\pi) = b_\pi \mathrm{I}_\pi$ for all $\pi \in \hat{G}$ such that

$$\hat{\mu}(\pi) = exp\,\{\lambda(b_\pi - 1)\mathrm{I}_\pi\}. \text{ --(6)}$$

We introduce here a class of central symmetric measures play significant role in deriving Laplace distribution and Poisson distribution function. Let ρ be a symmetric infinitely divisible probability measure on R. Then Levy-Khintchine formula we get

$$\int_R e^{iux}\rho(dx) = e^{-\eta(u)}$$

where

$$\eta(u) = \frac{1}{2}\sigma^2 u^2 + \int_{R-\{0\}}(1 - \cos(uy)v(dy)) \text{ -------------------(7)}$$

where $\sigma \geq 0$ and v is a symmetric Levy measure on $R - \{0\}$ i.e. a σ −finite Borel symmetric measure for which $\int R - \{0\} = \max\{1, |x|^2\}\, v(dx) < x$ as shown by C. Berg. For each $\pi \in \hat{G}$, let K_π be the Casimir operator acting in V_π. $K_\pi = -k_\pi \mathrm{I}_\pi$ where $k_\pi > 0$. We define a class of central symmetric probability measures on G as follows.

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$$\hat{\mu}(\pi) = e^{-\eta\left(k_\pi^{\frac{1}{2}}\right)}\mathrm{I}_\pi. \text{ ---(8)}$$

Lo-Ng criterion implies that a probability measure μ always exists with such a Fourier transform. In case μ satisfies the relation (8). μ is called a central symmetric probability measure on G induced by an infinitely divisible probability measure on R denoted by $\mu \in CID_R(G)$.

4. EXAMPLES

(i) (Gaussian measure) let us take $v = 0$ and so $c_\pi = \exp\{-1/2\,\sigma^2 k_\pi\}$. Gaussian measure is embeddable into the Brownian motion given by $\hat{\mu}_t = \exp\{-1/2\,\sigma^2 k_\pi\}$ for $t \geq 0$ extensively studied by both analysts and probabilitsts.

(ii) (The Laplace distribution on G). We take $\sigma = 0$,

$$v(dx) = \frac{exp\left\{-\frac{|x|}{\beta}\right\}}{|x|} \text{ (with } \beta > 0) \text{ and } c_\pi = (1 + \beta^2 k_\pi)^{-1}\text{ k}$$

We consider a central symmetric compound Poisson distribution $\mu_{\lambda\gamma}$ and the conditions under which $\mu_{\lambda\gamma} \in CID_R(G)$. Let us take $\sigma = 0$ and v to be a finite symmetric measure in (7) where

$$\eta(u) = \lambda \int_{R-\{0\}} (1 - \cos(uy)\hat{v}(dy))$$

$\lambda = v(R - \{0\})$ and $\hat{v}(\cdot) = 1/\lambda\, v(\cdot)$. For $\mu_{\lambda\gamma} \in CID_R(G)$ with this value of λ for which $b_\pi = g\left(k_\pi^{\frac{1}{2}}\right)$ in (6) where $g(u) = \int R \cos(ux)\, \hat{v}(dx)$.

(iii) It include stable-type distributions where $\sigma = 0$ and $v(dx) = \frac{b}{|x|^{1+\alpha}} dx$ where $b > 0$ and $0 < \alpha < 2$. In this case $C_\pi = exp\{-b^\alpha K_\pi^{\alpha/2}\}$. We consider here the relativistic Schrodinger distribution for $m > 0$ where $c_\pi = \mathrm{e}^{-\left(\sqrt{\mathrm{m}^2+\mathrm{k}_\pi}-\mathrm{m}\right)}$. It again has $\sigma = 0$.

5. (Random – Nikodym derivative and its embedding theorem)

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We assume here that g is compact semi-simple Lie group having Lie algebra g. Also, $\mu \in M(G)$ has a density $k \in L^1(G, R)$ if μ is absolutely continuous with respect to normalised Haar measure on G. We define k to be the Radon-Nikodym derivative $\frac{d\mu}{d\sigma}$.

If a density k exists for $\mu \in M_c(G)$ with $\hat{\mu}(\pi) = c_\pi \mathrm{I}_\pi$ and $k \in L^2(G, R)$ then it is in the form

$$k(\sigma) = \sum_{\pi\in\hat{G}} d_\pi \overline{c_\pi} x_\pi(\sigma) \quad \text{---(9)}$$

Let $\{x_1, x_2, x_3, \dots\dots\dots\dots, x_d\}$ be a basis for the Lie algebra g of lef-invariant vector fields. We define the Sobolev space $H_p(G)$ as follows:

$$H_p(G) = \{f \in L^2(G);\ x_{i1,} \dots\dots\dots x_{ik}\, f \in L^2(G); 1 \le k \le p, i_1 \dots\dots, i_k = 1, \dots\dots, d\}$$

It is a complex separable Hilbert space with associated norm

$$|||f|||_p^2 = \|f\|^2 + \sum_{i_1\dots\dots,i_k} \|x_{i1,} \dots\dots\dots x_{ik} f\|^2.$$

And its equivalent norm is given by

$$|||f|||_p^2 = \sum_{\pi\in\hat{G}} d_\pi (1 + k_\pi)^p\, tr\left(\hat{f}(\pi)\hat{f}(\pi)\right) \quad \text{--------------------------(10)}$$

where $\hat{f}(\pi) = \int G^\pi(\sigma^{-1}) f(\sigma) d\sigma$ is the Fourier transform M. Ruzansky has shown that $H_p(G)$ coincides with the usual Sobolev space on a manifold constructed using partitions of unity. In particular the Sobolev embedding theorem extends to

$$C^\infty(G) = \cap_{k\in N} H_k(G). \quad \text{---(11)}$$

6. THEOREM

Let $\mu \in M_c(G)$ with $\hat{\mu}(\pi) = c_\pi \mathrm{I}_\pi$ for all $\pi \in \hat{G}$.

(i) The measure μ has a square-integrable density if and only if

$\sum_{\pi\in\hat{G}} d_\pi |c_\pi|^2 < \infty$ ----------------------------------(12)

(ii) The measure μ has a continuous density if

$\sum_{\pi\in\hat{G}} d_\pi^{3/2} |c_\pi|^2 < \infty$. -----------------------------------(13)

(iii) The measure μ has a C^k density if

$\sum_{\pi\in\hat{G}} d_\pi^2 (1+k_\pi)^p |c_\pi|^2 < \infty$. -------------------------(14)

Where $2p > k + d/2$

Proof

Here μ has a non-trivial Gaussian component if η is such that $\sigma > 0$ in (7). We show that μ has a c^∞ −density for all $\sigma > 0$. Combine the relation (7), (11) and (14). Let us assume $\eta(u) \ge 1/2\, \sigma^2 u^2$ for all $u \in R$ to see that for all $k \in N$

$$\sum_{\pi\in\hat{G}} d_\pi^2 (1+k_\pi)^k c_\pi^2 \le \sum_{\pi\in\hat{G}} d_\pi^2 (1+k_\pi)^k exp\{-\sigma^2 k_\pi\}$$

$$\le M \sum_{\lambda\in P\cap D} |\lambda|^{2m} (1+|\lambda|^2)^K exp\ \{-\sigma^2 k_\pi\}$$

$$\le K_1 \sum_{n\in z^r} \|n\|^{2m} (1+\|n\|^2)^K exp\ \{-k_2\|n\|^2\}$$

$$= K_1 \sum_{j=0}^{\infty} a(j) j^m (1+j)^k\, exp\{-K_{2j}\}$$

$$\le K_1 \sum_{j=0}^{\infty} j^m \left(2\sqrt{j}+1\right)^r (1+j)^k\, exp\{-K_{2j}\} < \infty$$

where $M, K_1, K_2 > 0, a(j) = \#\{n \in Z^r; \|n\|^2 = j\}$

where $a(j) \le \left(2\sqrt{j}+1\right)^r$ for all $j \in N^2$.

7. THEOREM

For each $\lambda > 0$, R_λ is a pseudo-differential operator having symbol $(\lambda - L_\pi)^{-1}$ at $\pi \in \hat{G}$. If G is a compact Lie group and $\mu_{\mathrm{I}} \in CID_R(G)$ then for all $p \ge 2$, $H_p(G) \subseteq Dom(A)$ and A is a bounded linear operator form $H_p(G)$ to $H_{p-\mathrm{I}}(G)$.

Proof

We consider that there exists $K > 0$ such that $|\eta(u)| \le K(1+|u|^2)$ for all $u \in R$. Hence for each $f \in$ DOM(A)

$$\|Af\|_{p-1} = \sum_{\pi\in\hat{G}} d_\pi (1+K_\pi)^{p-1} \left\|\mathcal{L}_\pi \hat{f}(\pi)\right\|_{hs}^2$$

$$= \sum_{\pi\in\hat{G}} d_\pi (1+K_\pi)^{p-1} \left|\eta\left(K_\pi^{\frac{1}{2}}\right)\right|^2 \left\|\hat{f}(\pi)\right\|_{hs}^2$$

$$\le K \sum_{\pi\in\hat{G}} d_\pi (1+K_\pi)^p \left\|\hat{f}(\pi)\right\|_{hs}^2 \quad 6$$

$$= K\|f\|_p$$

Hence the theorem is proved.

CONCLUSION

The main thrust is on to compute the trace in both L^2 $-$space and the subspace of central functions.

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