PROPERTIES OF FUZZY UTILITY FUNCTIONS

Rani Begam

Univ. Dept. of Mathematics, T.M. Bhagalpur University, Bhagalpur.

Abstract : In this paper, we prove some theorems regarding the properties of a fuzzy utility function.

Keywords : fuzzy utility function, iso-membership, fuzzy utility indicator.

In this paper, we prove some theorems regarding the properties of a fuzzy utility function. For expositional purpose, we confine the analysis to a simple case of two commodities, labor service and wheat. Let X_1 represent the amount of labor supplied by a household and X_2 the amount of wheat available to the household.

Theorem 1: If every fuzzy utility indicator of a fuzzy utility relation is convex and normalized, the iso-membership grade surfaces (in Fig. 1) of the fuzzy utility relation are connective and never intersect each other.

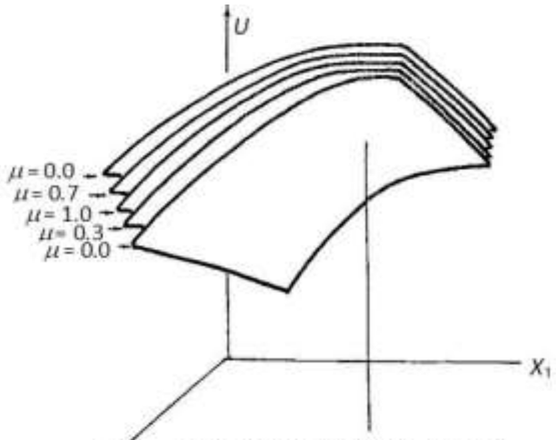

Fig. 1. Part of fuzzy utility function and its iso-membership grade surfaces.

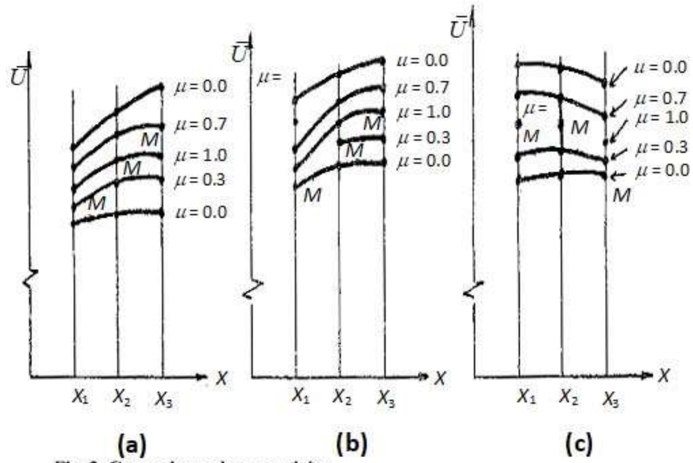

Fig. 2. Convexity and connectivity.

Proof : This theorem is apparent directly from the definition of convexity of fuzzy set. Several examples of iso-membership grade surfaces (here reduced to lines) for convex and normalized fuzzy utility indicators are shown in Figure 2(a). Figure 2(b) shows a case where one of the fuzzy utility indicators, M_1, is non-convex, and Figure 2(c) shows another case where one of them, M_2, is non-normalized. These cases lead to the existence of intersections between the iso-membership grade surfaces and/or the lack of connectiveness.

The lack of an intersection between the iso-membership grade surfaces in fuzzy utility relation further leads to the lack of intersections among the contours of α-cut in the fuzzy indifference set.

If the marginal utilities of every iso-membership grade surface of a fuzzy utility relation is diminishing, i.e., $\partial^2 f / \partial^2 x_i > 0$, then every contour of the α-cut of indifference set is concave in Figure 3. While diminishing marginal utilities or diminishing marginal rate of substitutions implies concavity of the indifference curve, diminishing marginal utility here implies convexity of iso-membership grade surface of a fuzzy utility relation. Figure 4 is used for the explanation of diminishing marginal rate of substitutions. Note that following the increase of x_1, the same increment Δ_{x_1} corresponds to a smaller increment Δ_{x_2}.

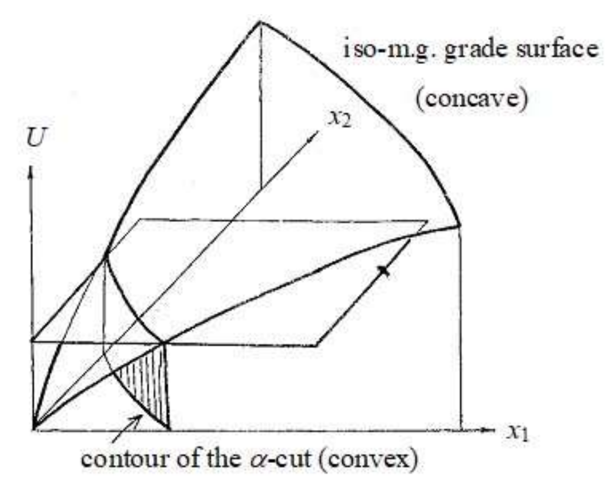

Fig. 3. Iso-membership grade surface and contour of the α-cut of an indifference set.

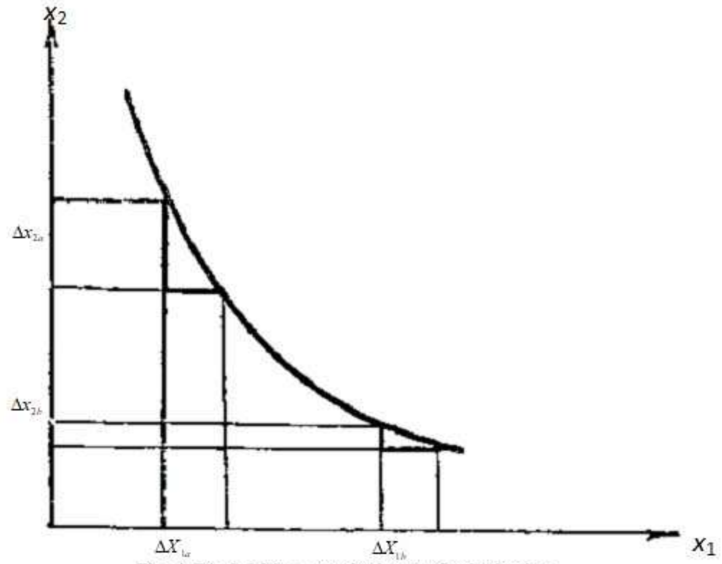

Fig. 4. Diminishing marginal rate of substitution.

Theorem 2: If every fuzzy utility indicator of a fuzzy utili[...]g marginal utilitie for every iso-membership grade surface of R_f then the weak M- preference set R_M, the weak u-preference set R_u

and the weak β-preference set R_β are all convex.

Proof : Since R_u and R_β are special cases of R_M, it suffices to prove the convexity of R_M. We delineate the procedure of constructing the weak preference set R_M first. According to Proposition 3, for obtaining the weak preference set, all we have to do is to make the composition operation between 1(M) and R_f^{-1}. The operations include the cylindrical extension operation of 1(M), the intersection operation between l(M) and R_f^{-1} and the projection operation of intersection set to commodity space X.

In Fig. 5, we construct the section labelled F of a fuzzy utility relation R_f^{-1} in any plane vertical to X_2. Cutting F by a line with equal utility level u, we obtain fuzzy set A on X_l. We assume that $u_l \in$ l(M) and its membership grade is $\mu_{1(M)}(u_1)$. The intersection of $\mu_A(x_l)$ and $\mu_{1(M)}(u_1)$ is a fuzzy set A'. We taking the union of all A' for all u_l in the support set of 1(M), we obtain all commodity bundles and their membership function values of the weak preference set below this vertical plane. Repeating this process for all $x_2 \in X_2$ we get the entire weak preference set.

Now, let us focus our attention on point a. For $\forall x \mid x \geq a$, we can always find an $u_2 > u_1$, for which

$$\mu_{R_f}{}^{-1}(x,u_2)(x) = 1$$

Due to the convexity property of 1(M), we have $\mu_{1(M)}(u_2) > \mu_{1(M)}(u_1)$. Thus

$$\mu_{1(M)}(u_2) \vee \mu_{R_f}{}^{-1}(x,u_2)(x) = \mu_{1(M)}(u_2) \geq \mu_{1(M)}(u_1) \geq \mu_A(x).$$

It means that we can always find a fuzzy set B' which is the component of weak preference set and which covers one specific point in the lagging edge of A'. The union operation in the process of constructing weak preference set will delete this specific point in A'. The lagging edge of A' is a part of the lagging edge of A. Another part of the lagging edge of A has already lost its effect when the intersection operation between A and $\mu_{1(M)}(u_1)$ was performed. Thus, the entire lagging edge of fuzzy set A fails to make any contribution to the construction of the weak preference set.

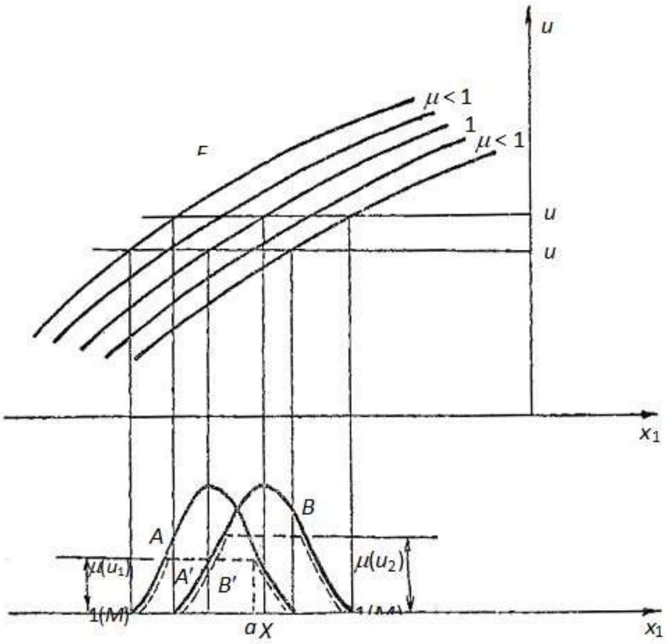

Fig. 5. The procedure of constructing the weak preference set.

Since this is true for every u, the membership grade of all the points in the consumption space under the surface in which the membership grade equals to 1 may be replaced by 1 without affecting the evaluation of the weak preference set.

From this replacement, we obtain a new F_f^{-1} denoted by R_f^{-1}. It is convex if every utility indicator of the determinate commodity bundle is convex and the marginal utility of every iso-membership surface of fuzzy utility function is diminishing.

Suppose that l(M) is assumed to be convex. Since the intersection of two convex fuzzy sets is convex, and the project of a convex fuzzy set is also convex.

REFERENCES

Allen, R. G. D. (1933). On the Marginal Utility of Money and Its Application. Economica New Series 13.
Dubois, D. and Prade, H., (1980). Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York.
Fisher, Irving (1927.) A Statistical Method for Measuring "Marginal Utility" and Testing the Justice of a Progressive Income Tax. In Economic Essays: Contributed in Honor of John Bates Clark, Edited by Jacob H. Hollander. New York: Macmillan.
Johnson, W. E. 1913 : The Pure Theory of Utility Curves. Economic Journal 23.
Samuelson, Paul A. (1952). Probability, Utility, and the Independence Axiom. Econometrica 20.