BI-PENDANT DOMINATION IN GRAPHS

Swaroopa Rani N C

Department of Mathematics

Government College (Autonomous), Mandya

Pavithra G B

Department of Mathematics ,

Government First Grade College, Pandavapura

Abstract

Let G be any graph. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The least cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. A pendant dominating set S of a graph G is a bi-pendant dominating set if $\langle V - S \rangle$ also contains a pendant vertex. The least cardinality of the bi-pendant dominating set in G is called the bi-pendant domination number of G, denoted by $\gamma_{bpe}(G)$. In this paper we study the properties of this parameter. The exact values for some families of standard graphs are obtained and also the bounds are estimated.

Keywords: Dominating set, Pendant Dominating set, Bi-pendant Dominating set

1.Introduction

Let $G = (V, E)$ be any graph with $|V(G)| = n$ and $|E(G)| = m$ edges. Then n, m are respectively called the order and the size of the graph G. For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N(v)$ containing v and all the vertices u adjacent to v. Let S be any subset of V, then the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

The minimum and maximum of the degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is said to be regular if $\delta(G) = \Delta(G)$. A vertex v of a graph G is called a cut vertex if its removal increases the number of components. A bridge or cut edge of a graph whose removal increases the number of components. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to pendant vertex is called a pendant edge. The graph containing no cycle is called a tree. A complete bi-partite graph $K_{1,3}$ is called a claw-free graph.

The corona of two disjoint graphs G_1 and G_2 is defined to be the graph $G = G_1 o\, G_2$ formed one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. If G and H are disjoint graphs, then the join of G and H denoted by $G + H$ is the graph such that $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup uv{:}\, u \in V(G), v \in V(H)$. The line graph $L(G)$ of a graph G is the graph whose vertex set corresponds to the edges of G such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges if G are adjacent. Any graph G with at least one bridge is called bridged graph. The n-barbell graph is the simple graph obtained by connecting two copies of a complete graph C_n to a singleton graph $K_1$with a bridge. The ladder graph is Cartesian product of P_2 and P_n where P_n is a path graph. The crown graph S_n for $n \geq 3$ is the graph with vertex set $V = \{u_1, u_2, \dots., u_n, v_1, v_2, \dots., v_n\}$ and an edge from $V = \{u_i,\ v_j, : 1 \leq i,\ j \leq n; i \neq j\}$. Therefore S_n coincides with the complete bipartite graph S_n with horizontal edges removed. The helm graph H_n is the graph obtained from n-wheel graph by joining a pendant edge at each node of the cycle. The helm graph H_n has $2n + 1$ vertices and 3n edges. The following definitions require for our study.

Definition 1.1. A subset S of V(G) is a dominating set of G if each vertex $u \in V - S$ is adjacent to a vertex in S. The least cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$.

Definition 1.2. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}(G)$.

2 The Bi-Pendant Domination Number of a Graph

Definition 2.1 A pendant dominating set S of a graph G is a bi-pendant dominating set if $\langle V - S \rangle$ also contains pendant vertex. The least cardinality of the bi-pendant dominating set is called the bi-pendant domination number of G, denoted by $\gamma_{bpe}(G)$.

The domination parameter is defined for all non-trivial connected graphs of order at least four. Hence, throughout the paper we assume that by a graph we mean a connected graph of order at least four.

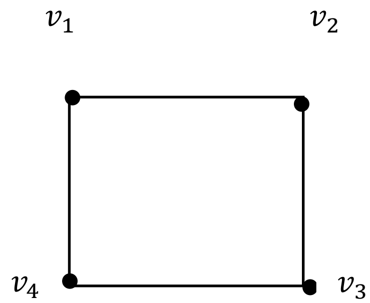

Figure 1: Cycle on 4 vertices

The bi-pendant domination number is not defined for the complete graph and bistar graph. In complete graph bi pendant domination is defined only $n = 4$. i.e., $\gamma_{bpe}(K_4) = 2$ in all other cases γ_{bpe} is not defined.

Observation 2.2. For any connected graph G with $n \geq 5, \gamma_{bpe}(G) \leq n - 1$.

Example 2.3. The possible minimum bi-pendant dominating sets for the following graph G are :

- **(i)** $\boldsymbol{D_1 = \{v_1, \ v_2\}}$
- **(ii)** $\boldsymbol{D_2 = \{v_2, \ v_3\}}$
- **(iii)** $\boldsymbol{D_3 = \{v_3, \ v_4\}}$
- **(iv)** $\boldsymbol{D_4 = \{v_4, \ v_1\}}$

Theorem 2.4. Let P_n be a path with $n \geq 5$ vertices. Then

$$\gamma_{bpe}(P_n) = \begin{cases} \frac{n}{3} + 1, & if\ n = 0\ (mod\ 3); \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n = 1\ (mod\ 3); \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n = 2\ (mod\ 3). \end{cases}$$

Proof: Let $G \cong P_n$ be a path and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We consider the following possible cases here:

Case 1:Suppose $n = 0\ (mod\ 3)$. Then $n = 3k$, for some integer $k > 1$. Then the set $S = \{v_2, v_3, v_{3i} | 2 \leq i \leq k\}$ is a bi-pendant dominating set of G. Hence $\gamma_{bpe}(G) \leq |S|$. i.e., $\gamma_{bpe}(G) = \frac{n}{3} + 1$. On the other hand, we have $\gamma(G) = \frac{n}{3}$ and any minimum dominating set of G contains only isolated vertices. Thus $\gamma_{bpe}(G) \geq \frac{n}{3} + 1$. Therefore, $i.e.,\ \gamma_{bpe}(G) = \frac{n}{3} + 1$.

Case 2: Suppose $n \equiv 1\ (mod\ 3)$. Then $n = 3k + 1$, for some integer $k > 1$. Then it is easy to check that any $\gamma -$ set S in G contains a pendant vertex and $\langle V - S \rangle$ also contains a pendant vertex. Hence any $\gamma -$ set S in G itself a bi-pendant dominating set in G. Therefore $\gamma_{bpe}(G) = \gamma(G) = \left\lceil \frac{n}{3} \right\rceil$

Case 3: Proof of this case is similar to Case 1.

Theorem 2.5. Let C_n be a cycle with n ≥ 4 vertices. Then

$$\gamma_{bpe}(C_n) = \begin{cases} \frac{n}{3} + 1, & if\ n = 0\ (mod\ 3); \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n = 1\ (mod\ 3); \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n = 2\ (mod\ 3). \end{cases}$$

Observation 2.6. Let G be a ladder graph with 2n vertices. Then $\gamma_{bpe}(G) = \left\lceil \frac{n}{2} \right\rceil + 1$

Proposition 2.7. For any helm graph H_n with $2n + 1$ vertices. Then $\gamma_{bpe}(H_n) = n + 1$

Proof : Let (XY) be a partition of H_n with $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_n\} \cup \{v\}$. Where $\{v\}$ is the vertex attached to all vertices in the set Y. Let v, $u_1$are the two adjacent vertices of the graph H_n and S is the set of all collection of leaves of H_n, except the leaf of u_1. Then the set $S' = |S| \cup \{v, u_1\}$ will be a bi-pendant dominating set of H_n. Therefore $\gamma_{bpe}(H_n) = |S'| = (n - 1) + 2 = n + 1$.

Theorem 2.8. Let G be a wheel graph with n vertices and n≥ 3. Then $\gamma_{bpe}(W_n) = 2$.

Proof : Let G be a wheel graph of order $n \geq 3. Then\ G \cong C_{n-1} + K_1$. The set $S = \{u, v\}$ is pendant dominating set of G where v is the vertex in K_1 and $u \in C_{n-1}$. Therefore S is itself a bi-pendant dominating set of G. Hence $\gamma_{bpe}(W_n) = |S| = 2$

Observation 2.9. Let G be a crown graph with 2n vertices. Then $\gamma_{bpe}(G) = n$

Theorem 2.10. Let $G \cong K_{m,n}$be a complete bipartite graph with $m \leq n$. Then $\gamma_{bpe}(K_{m,n}) = m$

Proof: Let $G \cong K_{m,n}$ be a complete bipartite graph with $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V_1 = \{u_1, u_2, \dots, u_m\}$ are two partite set in G. The bi-pendent dominating set of G is obtained by taking the one vertex in partite set V_1 and m-1 vertices in the another partite set V_2. Therefore

$\gamma_{bpe}(G) = 1 + (m - 1) = m.$

Theorem 2.11. Let G be a barbell graph of order n . Then $\gamma_{bpe}(G) = n - 1$.

Proof : Let G be a barbell graph and let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let v_1 and v_2 be the adjacent vertices of G is attached to the copies of complete graph. The bi-pendent dominating set of G is obtained by taking the vertices v_1, $v_2$and (n-3) vertices in any one copies of complete graph . Therefore $\gamma_{bpe}(G) = 2 + (n - 3) = n - 1$.

Theorem 2.12. Let G be a pan Graph .Then $\gamma_{bpe}(G) = 2 + \left\lceil\frac{n-3}{3}\right\rceil$.

Proof. Let G be a pan graph with vertices $\{v_1, v_2, \dots, v_n\}$ where v_n is the vertex attached to the vertex v_1 of C_n. Fix an edge $e = v_1v_n$. Then $\gamma_{bpe}(G) = \{u, v\} \cup \gamma(H)$ where H is the graph obtained by removing the vertices v_1, v_n and its neighbor from G. Clearly $H \cong P_{n-3}$. Hence $\gamma_{bpe}(G) = 2 + \gamma P_{n-3} = 2 + \left\lceil\frac{n-3}{3}\right\rceil$.

Theorem 2.13. If G is a graph then $\gamma_{bpe}(G) = 2$ if and only if $G \cong T + K_1$. Where T is a tree of order $n \geq 3$.

Proof. Assume that $G \cong T + K_1$, then clearly the set $S = \{u, v\}$ will be a bi-pendant dominating set of G. where u and n are vertices in T and K_1 respectively.

Conversely, if $\gamma_{bpe}(G) = 2$ then there exist a bi-pendant dominating set of G with $|S| = 2$. Such that $\langle V - S \rangle$ is a tree. Since each vertex in $\langle V - S \rangle$ is adjacent to the vertex in S.

Let $\mathcal{G}$ be the collection of graphs o following types. A cycle, complete graph of order 4, cycle, path and wheel of order 5 and $K_{2,2}$.

Theorem 2.14. Let G be a connected graph of order n. Then $\gamma_{bpe}(G) = n - 2$ if and ony if

$G \in \mathcal{G}$

Theorem 2.15. For any integer $a > 0$, there exist a connected graph G such that $\gamma(G) = \gamma_{bpe}(G) = a + 1$

Proof. Let $P_j : \{u_j, v_j, w_j, x_j, y_j\}(1 \leq j \leq a)$ be a path of order 5. Let G be a graph obtained from P_j $(1 \leq j \leq a)$ by adding new vertex X and joining X with u_j $(1 \leq j \leq a)$, v_j $(1 \leq j \leq a)$, w_j $(1 \leq j \leq a)$, and y_j $(1 \leq j \leq a)$. The graph G is shown in figure 2.

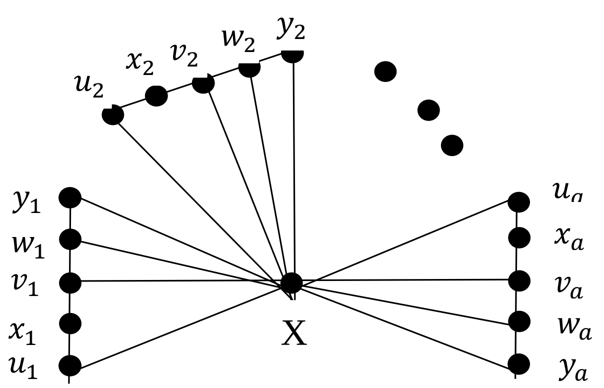

Figure 2

We show that $\gamma(G) = \gamma_{bpe}(G) = a + 1$. $H_j = \{v_j, u_j, x_j\}(1 \leq j \leq a)$. Its easily observed that X belongs to every minimum bi-pendant dominating se of G and so $\gamma_{bpe}(G) > 1$. Also its easily seen that every dominating set of G contains at least one element of H_j $(1 \leq j \leq a)$ and so $\gamma_{bpe}(G) \geq a + 1$. Now the set $S = \{X\} \cup \{v_1, v_2, \ldots, v_a\}$ will be a bi-pendant dominating set G. So that $\gamma(G) = \gamma_{bpe}(G) = a + 1$

Proposition 2.16. Let G be any graph with n vertices. Then $\gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{bpe}(G)$. Equality holds if G is a cycle of order 4.

Proof. Since every bi-pendant dominating set is a pendant dominating set and every pendant dominating set is a dominating set of G, it follows that $\gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{bpe}(G)$. Suppose G is a cycle with 4 vertices. Then $\gamma(G) = \gamma_{pe}(G) = \gamma_{bpe}(G) = 2$.

Theorem 2.17. For any graph G, we have $\gamma(G) \leq \gamma_{bpe}(G) \leq \gamma(G) + \delta(G)$.

Proof. Since a bi-pendant dominating set of G is a dominating set, it follows that $\gamma(G) \leq \gamma_{bpe}(G)$. Now let v be a vertex in G with $\deg(v) = \delta(G)$ and let S be a dominating set in G contains N[v] so that the set $S' = S \cup N[v]$ will be a bi-pendant dominating set of G, it follows that $\gamma_{bpe}(G) \leq \gamma(G) + \delta(G)$ and hence the right equality follows.

Proposition 2.18. Let G be a graph with n vertices. Then $\gamma(G) + \gamma_{bpe}(G) \leq n$.

Proof. Let S be a bi-pendant dominating set. Then S is a dominating set and $\langle V - S \rangle$ contains a pendant vertex. Obviously, $\gamma_{bpe}(G) \leq |S|$. Since S is a dominating $\langle V - S \rangle$ is also a dominating. Thus $\gamma(G) \leq |V - S|$. Hence $\gamma(G) \leq \gamma_{bpe}(G) \leq |S| + |V - S| = n$, proving the result.

Theorem 2.19. Let G be a connected graph with n vertices and H be any graph. Then $\gamma_{bpe}(G\ o\ H) = \begin{cases} n, & if\ \delta(H) = 1 \\ n + 1, & otherwise \end{cases}$

Proof. For any connected graph with n vertices and H be any graph, we have $\gamma(G\ o\ H) = n$ and hence $\gamma_{bpe}(G\ o\ H) \leq n + 1$. First, suppose H has a pendant vertex, then clearly the set $S = |V(G)|$ is a bi-pendant dominating set in $(G\ o\ H)$. If $\delta(H) \geq 2$, then the set $S = |V(G)| \cup \{u\}$ will be a bi-pendant dominating set of $G\ o\ H$, where u is a vertex in H is adjacent to any one vertex in G. Therefore $\gamma_{bpe}(G\ o\ H) = |S| = n + 1$

Theorem 2.20. Let G be any graph. If $diam(G) \geq 3$ then $\gamma_{bpe}(\bar{G}) = 2\ or\ 3$

Proof. If G has a pendant vertex the clearly $\gamma_{bpe}(G) = 2$. Let G be a connected graph of diameter at least 3. If $u, v \in V(G)$ with $diam(u, v) \geq 3$ then the set $S = \{u.v\}$ is a pendant dominating set of $\bar{G}$. The bi-pendant dominating set of $\bar{G}$ is obtained by chosing any one vertex nonadjacent to the vertices $\{u, v\}$ together with the pendant dominating set of $\bar{G}$. Therefore $\gamma_{bpe}(\bar{G}) = 3$

Theorem 2.21. Let G be a triangle free graph order at least3. Then $\gamma_{bpe}(\bar{G}) = 2\ or\ 3$

Proof. Let G be a triangle free graph. If G contains a pendant vertex and an isolated vertex then clearly $\gamma_{bpe}(\bar{G}) = 2$. Suppose G has no pendant and an isolated vertex, then G contain atleast one edge say $e = uv$. As G is triangle free no vertex in G can be adjacent to both u and v. Thus $S = \{u, v\}$ will be a γ_{pe} – set in $\bar{G}$. Now, for any vertex $w \in V(G)$, the set $S \cup \{w\}$ will be a γ_{bpe} – set in $\bar{G}$. Hence $\gamma_{bpe}(\bar{G}) = 3$.

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