A NOTE ON FIBONACCI SEQUENCE

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Abstract: *One of the famous sequence in mathematics is Fibonacci sequence. Many books and papers are published on Fibonacci sequence. In this article we discuss about some important results on Fibonacci sequence.*

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Introduction: The Italian mathematician Leonardo de Pisa was born in Pisa around 1175 AD. He is commonly known as Fibonacci which is a shortened form of Filius Bonaccio (son of Bonaccio). His father, Bonaccio, was a customs inspector in the city of Bugia on the north coast of Africa (presently Bougie in Algeria) and as a result, Fibonacci was educated by the Mohammedans of Barbary. He was taught the Arabic system of numbers and in the early thirteenth century returned to Italy to publish the book ***Liber Abaci*** (Book ***offhe Abacus)*** in **1202** (Leonardo di Pisa **1857).** This book introduced the Arabic system of numbers to Europe and is responsible for Fibonacci's reputation as the most accomplished mathematician of the middle ages. The book also posed a problem involving the progeny of a single pair of rabbits which is the basis of the Fibonacci sequence (or Fibonacci series).

The **rabbit** problem is as follows:

A pair of adult rabbits produces a pair of baby rabbits once each month. Each pair of baby rabbits requires one month to grow to be adults and subsequently produces one pair of baby rabbits each month thereafter. Determine the number of pairs of adult and baby rabbits after some number of months. It is also assumed that rabbits are immortal.

This problem may be expressed mathematically in this way: The number of adult rabbit pairs in a particular month (say month $n+2$)*, F_{n+2},* is given by the number of adult rabbit pairs in the previous month, A_{n+1}*,* plus the number of baby rabbit pairs from the previous month which grow to be adults, b_{n+1};

$$F_{n+2} = F_{n+1} + b_{n+1} \qquad\qquad (1)$$

In a given month (say month $n+1$), the number of pairs of baby rabbits will be equal to the number of adult rabbit pairs in the previous month;

$$b_{n+1} = F_n \qquad\qquad (2)$$

Combining (1) and (2) gives the recursion relation for the number of adult rabbit pairs as

$$F_{n+2} = F_{n+1} + F_n \qquad\qquad (3)$$

Growth of Rabbit

Months	Adult Pairs	Young Pairs	Total
1	1	1	2
2	2	1	3
3	3	2	5
4	5	3	8
5	8	5	13
6	13	8	21
7	21	13	34
8	34	21	55
9	55	34	89
10	89	55	144

Hence the recursive relation (3) gives a sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … with F_1 = 1, F_2 = 1 and $F_{n+2} = F_{n+1} + F_n$. This sequence is called Fibonacci sequence.

The Fibonacci sequence grows rapidly.

Theorem 1: We have

$$F_{5n+2} > 10^n \quad \text{for} \quad n \geq 1.$$

Proof: Proof is by induction. For $n = 1$, we have $A_7 = 13 > 10$. Let us assume that the inequality holds for arbitrary integer $n = k$, that is $F_{5k+2} > 10^k$. Now we show that it also holds for $n = k + 1$. Using the recursive formula (3) several times, we can write

$$\begin{aligned}
F_{5k+7} &= 8F_{5k+2} + 5F_{5k+1} \\
&> 8F_{5k+2} + 2(F_{5k+1} + F_{5k}) \\
&= 10F_{5k+2} > 10 \cdot 10^k = 10^{k+1}
\end{aligned}$$

completing the induction step and the argument.

Theorem 2: For the Fibonacci sequence, $\gcd(F_{n+1}, F_n) = 1$ for every $n \geq 1$.

Proof: Suppose $\gcd(F_{n+1}, F_n) = d$. Then d / F_{n+1} and d / F_n and hence $d / F_{n+1} - F_n$ i.e, d / F_{n-1}. Now d / F_n and d / F_{n-1} implies d / F_{n-2}. This shows d divides all Fibonacci numbers $F_{n-2}, F_{n-3}, …$ and hence d / F_1. But $F_1 = 1$. Hence $d = 1$. This completes the proof.

We next prove a very important Theorem that the greatest common divisor of two Fibonacci numbers is itself a Fibonacci number. In fact $\gcd(F_n, F_m) = u_d$ where $d = \gcd(n, m)$. Before proving this Theorem, we prove the following Theorems.

Theorem 3: We have

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1} \qquad (4)$$

Proof: We prove it by induction on n. When $n = 1$, (4) takes the form

$$F_{m+1} = F_{m-1}F_1 + F_mF_2 = F_{m-1} + F_m$$

which is obviously true. Let us assume that the formula holds for $n = 1, 2, 3, \ldots, k$ and try to verify it when $n = k + 1$. By our assumption, we have

$$F_{m+k} = F_{m-1}F_k + F_mF_{k+1}$$
$$F_{m+(k-1)} = F_{m-1}F_{k-1} + F_mF_k.$$

Addition of these two equations gives us

$$F_{m+k} + F_{m+(k-1)} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k).$$

From (3), we have

$$F_{m+(k+1)} = F_{m-1}F_{k+1} + F_mF_{k+2}$$

Which is precisely (4) with n replaced by $k + 1$. Hence, by induction (4) holds for all $m \geq 2$ and $n \geq 1$.

Theorem 4: For $m \geq 1$ and $n \geq 1$, F_{mn} is divisible by F_m.

Proof: Proof is by induction on n. The result is obviously true when $n = 1$. Let us assume that F_{mn} is divisible by F_m for $n = 1, 2, \ldots, k$. From (4), we have

$$F_{m(k+1)} = F_{mk-1}F_m + F_{mk}F_{m+1}$$

Because F_m divides F_{mk} by supposition, the right hand side of this expression (and hence, the left hand side) must be divisible by F_m. Accordingly, $F_m / F_{m(k+1)}$, which was to be proved.

Lemma 1: If $m = qn + r$, then $\gcd(F_m, F_n) = \gcd(F_r, F_n)$.

Proof: From (4), we have

$$\gcd(F_m, F_n) = \gcd(F_{qn+r}, F_n) = \gcd(F_{qn-1}F_r + F_{qn}F_{r+1}, F_n).$$

Now from Theorem 4 and the fact that $\gcd(a+c, b) = \gcd(a, b)$, whenever b / c, gives

$$\gcd(F_{qn-1}F_r + F_{qn}F_{r+1}, F_n) = \gcd(F_{qn-1}F_r, F_n).$$

We claim that, $\gcd(F_{qn-1}F_r, F_n) = 1$. Suppose $d = \gcd(F_{qn-1}F_r, F_n)$. Then d / F_n and F_n / F_{qn} implies d / F_{qn} and therefore d is a common divisor of the successive Fibonacci number F_{qn-1} and F_{qn}. Because successive Fibonacci numbers are relatively prime, $d = 1$.

We know that if $\gcd(a, c) = 1$, then $\gcd(a, bc) = \gcd(a, b)$. Hence, we have

$$\gcd(F_m, F_n) = \gcd(F_{qn-1}F_r, F_n) = \gcd(F_r, F_n).$$

Let us now prove our main theorem.

Theorem 5: The greatest common divisor of two Fibonacci numbers is again a Fibonacci number, that is,

$$\gcd(F_n, F_m) = u_d \text{ where } d = \gcd(n, m).$$

Proof: Assume that $m \geq n$. Applying the Euclidean Algorithm to m and n, we get the following system of equations:

$$m = q_1 n + r_1 \qquad\qquad 0 < r_1 < n$$
$$n = q_2 r_1 + r_2 \qquad\qquad 0 < r_2 < r_1$$
$$r_1 = q_3 r_2 + r_3 \qquad\qquad 0 < r_3 < r_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$r_{n-2} = q_n r_{n-1} + r_n \qquad\qquad 0 < r_n < r_{n-1}$$
$$r_{n-1} = q_{n+1} r_n + 0.$$

Now, from above Lemma 1, we have

$$\gcd(F_m, F_n) = \gcd(F_{r_1}, F_n) = \gcd(F_{r_1}, F_{r_2}) = \cdots = \gcd(F_{r_{n-1}}, F_{r_n}).$$

Since r_n / r_{n-1}, by Theorem 4, we have $F_{r_n} | F_{r_{n-1}}$ and hence gcd $(F_{r_n}, F_{r_{n-1}}) = F_{r_n}$. But r_n, being the last nonzero remainder in the Euclidean Algorithm for m and n, is equal to gcd(m, n). Hence $\gcd(F_n, F_m) = F_{\gcd(m,n)}$. This completes the proof.

Corollary 1: In the Fibonacci sequence, F_m / F_n if and only if m / n for $n \geq m \geq 3$.

Proof: One way we have already proved in Theorem 4. So, assume that F_m / F_n. Then $\gcd(F_m, F_n) = F_m$. But, by Theorem 5, the value of $\gcd(F_m, F_n) = F_{\gcd(m, n)}$. Hence, $\gcd(m, n) = m$ and therefore m / n.

Example 1: We show that $\gcd(F_{16}, F_{12}) = \gcd(987, 144)$. From Euclidean algorithm, we have gcd(987, 144) = 3 and gcd(16, 12) = 4. Now, $F_4 = 3$. Hence $\gcd(F_{16}, F_{12}) = F_{\gcd(16, 12)}$.

Certain Identities Involving Fibonacci Numbers

We move on and develop several of the basic identities involving Fibonacci numbers; these should be useful in doing the problem at the end of the section. One of the simplest asserts that the sum of the first n Fibonacci numbers is equal to $F_{n+2} - 1$. For instance, when the first eight Fibonacci numbers are added together, we obtain

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54 = 55 - 1 = F_{10} - 1$$

That this is typical of the general situation follows by adding the relations

$$F_1 = F_3 - F_2$$
$$F_2 = F_4 - F_3$$

$$F_3 = F_5 - F_4$$

$$\vdots$$

$$F_{n-1} = F_{n+1} - F_n$$

$$F_n = F_{n+2} - F_{n+1}$$

On doing so, the left-hand side yields the sum of the first n Fibonacci numbers, whereas on the right-hand side the terms cancel in pairs leaving only $u_{n+2} - u_2$. But $u_2 = 1$. The consequence is that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1 \qquad\qquad (5)$$

Next, we prove another important identity:

Theorem 6: We have

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}. \tag{6}$$

Proof: We have

$$F_n^2 - F_{n+1}F_{n-1} = F_n(F_{n-1} + F_{n-2}) - F_{n+1}F_{n-1}$$
$$= (F_n - F_{n+1})F_{n-1} + F_n F_{n-2}.$$

From (3), this can be written as

$$F_n^2 - F_{n+1}F_{n-1} = (-1)(F_{n-1}^2 - F_n F_{n-2}).$$

The important point is that except for the initial sign the right-hand side of this equation is the same as the left-hand side, but with all the subscripts decreased by 1. By repeating the argument $(F_{n-1}^2 - F_n F_{n-2})$ can be shown to be equal to the expression $(-1)(F_{n-2}^2 - F_{n-1}F_{n-3})$, and hence

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^2(F_{n-2}^2 - F_{n-1}F_{n-3}).$$

Continuing this process, after $n - 2$ such steps, we arrive at

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-2}(F_2^2 - F_3 F_1)$$
$$= (-1)^{n-2}(1^2 - 2 \cdot 1) = (-1)^{n-1}.$$

This completes the proof.

Note 1: For n = 2k, (6) becomes $F_{2k}^2 = F_{2k+1}F_{2k-1} - 1$.

In 1843, the French mathematician Jacques-Philippe-Marie Binet discovered a formula for expressing F_n in terms of the integer n; namely,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

This formula can be obtained by considering the two roots

$$\tau = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \varphi = \frac{1-\sqrt{5}}{2} \tag{7}$$

of the equation $x^2 - x - 1 = 0$. Since they are roots of this equation, they must satisfy

$$\tau^2 = \tau + 1 \quad \text{and} \quad \varphi^2 = \varphi + 1.$$

Multiply first relation by τ^n and second by φ^n. Then, we obtain

$$\tau^{n+2} = \tau^{n+1} + \tau^n \quad \text{and} \quad \varphi^{n+2} = \varphi^{n+1} + \varphi^n.$$

Subtracting the second equation from the first, and dividing by $\tau - \varphi$, leads to

$$\frac{\tau^{n+2} - \varphi^{n+2}}{\tau - \varphi} = \frac{\tau^{n+1} - \varphi^{n+1}}{\tau - \varphi} + \frac{\tau^n - \varphi^n}{\tau - \varphi} \tag{8}$$

If we put $H_n = (\tau^{n+2} - \varphi^{n+2})/(\tau - \varphi)$, (8) can be written as

$$H_{n+2} = H_{n+1} + H_n \qquad n \geq 1.$$

From (7), we have

$$\tau + \varphi = 1 \qquad \tau - \varphi = \sqrt{5} \qquad \tau\varphi = -1$$

Hence,

$$H_1 = \frac{\tau - \varphi}{\tau - \varphi} = 1 \qquad H_2 = \frac{\tau^2 - \varphi^2}{\tau - \varphi} = \tau + \varphi = 1.$$

All this shows that the sequence $H_1, H_2, H_3, \ldots$ is precisely the Fibonacci sequence, which gives

$$F_n = \frac{\tau^n - \varphi^n}{\tau - \varphi} \qquad n \geq 1. \tag{9}$$

Note 2: The number $\tau = \dfrac{1 + \sqrt{5}}{2}$ is usually called **Golden Ratio.**

Note 3: (9) is called Binet's formula.

Using (9), we can prove certain identities involving Fibonacci numbers.

Theorem 7: We have

$$F_{n+2}^2 - F_n^2 = F_{2n+2}.$$

Proof: As we start, recall that $\tau\varphi = -1$ which has the immediate consequence $(\tau\varphi)^{2k} = 1$ for $k \geq 1$, Then,

$$
\begin{aligned}
F_{n+2}^2 - F_n^2 &= \left(\frac{\tau^{n+2} - \varphi^{n+2}}{\tau - \varphi}\right)^2 - \left(\frac{\tau^n - \varphi^n}{\tau - \varphi}\right)^2 \\
&= \frac{\tau^{2(n+2)} - 2 + \varphi^{2(n+2)}}{(\tau - \varphi)^2} - \frac{\tau^{2n} - 2 + \varphi^{2n}}{(\tau - \varphi)^2} \\
&= \frac{\tau^{2(n+2)} + \varphi^{2(n+2)} - \tau^{2n} - \varphi^{2n}}{(\tau - \varphi)^2}
\end{aligned}
$$

Now the expression in the numerator may be written as

$$\tau^{2(n+2)} - (\tau\varphi)^2 \tau^{2n} - (\tau\varphi)^2 \varphi^{2n} + \varphi^{2(n+2)} = (\tau^2 - \varphi^2)(\tau^{2n+2} - \varphi^{2n+2})$$

On doing so, we get

$$
\begin{aligned}
F_{n+2}^2 - F_n^2 &= \frac{(\tau^2 - \varphi^2)(\tau^{2n+2} - \varphi^{2n+2})}{(\tau - \varphi)^2} \\
&= (\tau + \varphi)\left(\frac{\tau^{2n+2} - \varphi^{2n+2}}{\tau - \varphi}\right) \\
&= 1.F_{2n+2} = F_{2n+2}.
\end{aligned}
$$

For a second illustration of the usefulness of the Binet formula, let us once again derive the relation $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$, (see Note 1). First, we calculate

$$F_{2n+1}F_{2n-1} - 1 = \left(\frac{\tau^{2n+1} - \varphi^{2n+1}}{\sqrt{5}}\right)\left(\frac{\tau^{2n-1} - \varphi^{2n-1}}{\sqrt{5}}\right) - 1$$

$$= \frac{1}{5}(\tau^{4n} + \varphi^{4n} - (\tau\varphi)^{2n-1}\tau^2 - (\tau\varphi)^{2n-1}\varphi^2 - 5)$$

$$= \frac{1}{5}(\tau^{4n} + \varphi^{4n} + (\tau^2 + \varphi^2) - 5).$$

Because $\tau^2 + \varphi^2 = 3$, this last expression becomes

$$\frac{1}{5}(\tau^{4n} + \varphi^{4n} - 2) = \frac{1}{5}(\tau^{4n} + \varphi^{4n} - 2(\tau\varphi)^{2n})$$

$$= \left(\frac{\tau^{2n} + \varphi^{2n}}{\sqrt{5}}\right)^2 = F_{2n}^2$$

which is the required identity.

Keywords: Fibonnaci sequence, identity, golden ratio

References:

(a)   Elementary Number Theory, David M. Burton, McGraw Hill Publication

(b)   The Golden Ratio and Fibonacci Numbers, Richard A. Dunlap, World Scientific.