NON-ASSOCIATIVE AND NON-DISTRIBUTIVE PARARING

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ABSTRACT : In this paper a particular type of associative (non-associative) non-distributive ring has been introduced with the help of a newly defined distributive property called the para-distributive property. Different types of pararing have been illustrated with examples. Lastly some important properties of the pararing have been studied.

Keywords: Para-distributive property, Basal, Mixed ring, Pararing, Sub-pararing.

1. Introduction

Non-associative and non-distributive rings of different varieties have introduced quasi-ring $R(+, 0)$ with the help of quasi-distributive properties [3] :

$x \text{ o} + (y + z) = x \text{ o } y + x \text{ o } z - x;$
$(x + y) \text{ o } z = x \text{ o } z + y \text{ o } z - z;$
$x, y, z \in R(+, 0).$

Quasi-distributive ring is not a ring and vice versa. Further, $\theta x \neq \theta$ and $x\theta \neq \theta$, θ being the nul1element of $R(+, 0)$.

$x \text{ o } (y + z) = x \text{ o } y = x \text{ o } z - x \text{ o } \theta;$ (1.1)
$(y + z) \text{ o } x = y \text{ o } x + z \text{ o } x - \theta \text{ o } x$ (1.2)

to deal with weak-ring and weak lie algebras respectively. Weak-ring and weak lie algebras are not ring and lie algebras respectively [l].

For quasi-vector space v over F we have used a generalised form of scalar distribution property

$(\alpha + \beta + \gamma)x = \alpha x + \beta x + \gamma x;\ \alpha, \beta, \gamma \in F, x \in v.$

In this paper para-distributive and right (left) para-distributive properties have been introduced in an analogous way leading to the definition of pararing, right (left) ring and mixed ring of a pararing such that any ring is a right (left) ring and mixed ring which are again pararing but the converse is not true.

2. Definitions.

(i) A system of double composition $R(+, 0)$ is said to satisfy para-distributive law and right (left) para-distrubitive law if

$x \text{ o } (y + z + \omega) = x \text{ o } y + x \text{ o } z + x \text{ o } \omega,$ (2.1)
$(x + y + z) \text{ o } \omega = x \text{ o } \omega + y \text{ o } \omega + z \text{ o } \omega,$ (2.2)
$x \text{ o } (y + z) = x \text{ o } y + x \text{ o } z;$ (2.3)
$(x + y + z) \text{ o } \omega = x \text{ o } \omega + y \text{ o } \omega + z \text{ o } \omega,$ (2.4)
$x \text{ o } (y + z + \omega) = x \text{ o } y + x \text{ o } z + x \text{ o } \omega,$ (2.5)
$(x + y) \text{ o } \omega = x \text{ o } \omega + y \text{ o } \omega.$ (2.6)

Non-distributive property (1.1) and (1.2) follow as a particular case of (2.1) and (2.2).

A system $R(+, 0)$, which is additively a group, is called a pararing or a left (right) ring of a pararing according as it satisfies (2.1) to (2.6).

The system $R(+, 0)$ is called a mixed ring if it is the union of a right and a let ring.

For a left and a right ring $\theta x \neq \theta$ and $x\theta \neq \theta$ respectively. For an associative mixed ring $\theta x\theta = \theta$. For a pararing $\theta x \neq \theta$, $x\theta \neq \theta$ and also $\theta x\theta \neq \theta$ if the pararing is associative.

(ii) A system $R(+, 0)$ is said to satisfy quasi-anticommutative property if

$x \text{ o } y + y \text{ o } x + y \text{ o } z + z \text{ o } y + z \text{ o } x + x \text{ o } z = \theta$

(iii) A pararing is called a Jordan pararing if it satisfies

$(x \text{ o } y) \text{ o } (x \text{ o } x) = [x \text{ o } \{y \text{ o } (x \text{ o } x)\}] = \theta$

(iv) A non-distributive system is said to satisfy binary non-lie properties of type

$x^2 = \theta, (((x \text{ o } y) \text{ o } y) \text{ o } x) + (((y \text{ o } x) \text{ o } x) \text{ o } y) = \theta.$
$x^2 = \theta, (((x \text{ o } y) \text{ o } x) \text{ o } y) + (((y \text{ o } x) \text{ o } x) \text{ o } y) = \theta.$

(v) A subset Y of a pararing X is called a subpararing of X if Y is a pararing with respect to the operation of x restricted to y.

(vi) A ring, left (right) ring and mixed ring of pararing X are all subpararing of X.

(vii) Y is called semi-basal if it is a mixed ring of X. Y is called basal if it is a ring of X. It is called X non-basal if it is not semi-basal.

(viii) A left (right) subpararing of a right (left) ring X is considered as pararing basal if it is a ring of X. It is called non-basal if it is not basal.

Examples: Let $R(+, \cdot)$ be a ring which is additively periodic of order 2 and let

(1) $x \text{ o } y = x + y + xy;$
(2) $x \text{ o } y = x + y + xy - yx;$
(3) $x \text{ o } y = x + y + x(xy) - x(yx).$

Thus $R(+, 0)$ is a pararing in each case.

3. Observation

(i) The above Example (1) is a Jordan pararing. It is, in general, non-associative. It does not satisfy above alternative Example and flexible laws. It is associative if $R(+, \cdot)$ is so. It does not satisfy $x^2 = \theta$, but it is quasi-anticommutative, where $R(+, \cdot)$ is commutative [2].

(ii) Example (2) is a pararing which satisfies binary non-lie property.

(iii) Example (3) is a pararing quasi-anticommutative but not anticommutative.

1. Let $G(+)$ be a group periodic of order 2. Then $G(+, 0)$ is a right ring if $x \text{ o } y = x$ and left ring if $x \text{ o } y = x$; $x, y \in G$.
2. Let $R(+, \cdot)$ be a ring which is additively periodic of order 2. Then $R(+, 0)$ is a right ring if
   $x \text{ o } y = y = x + xy$;
   $x \text{ o } y = x + xy + yx$.
   $R(+, 0)$ is a left ring if
   $x \text{ o } y = y + xy$;
   $x \text{ o } y = y + xy + yx$.

Let X be an arbitrary pararing. Let X' and X'' be a right (left) ring of X. Then

$X'_0 = \{x : x \in X',\ x\theta = 0\}$ is a basal subring of X';

$X''_0 = \{x : x \in X'', \theta x = \theta\}$ is a basal subring of X'';

$\bar{X}_{0,0} = \{x : x \in X, \theta x = \theta, x\theta = \theta\}$ is a basal subring of X.

If, further, X is associative, then

$X_{0,0} = \{x : x \in X, \theta x\theta = 0\}$ is a semi-basal subring of X.

Let X, X' and X'' are as defined in Section 4. Then

$X'^{,0} = \{x : x \in x'\}$ is a left subring of X';

$X''^{,0} = \{x : x \in x''\}$ is a right subring of X'';

$X = \{\theta x\theta, x \in x\}$ is a subring of X.

where $x\theta \cdot y\theta = (xy)\theta$, $\theta x \cdot \theta y = \theta(xy)$; $(\theta x\theta \cdot (\theta y\theta) = \theta(xy)\theta$.

4. Properties

Property 4.1. Let (X', X'') be a right (left) ring of an associative pararing X containing identity for multiplication. Then

(i) $(x', x'^{,0}, x'_0)$ is additively commutative iff
$(x + x'^{,0}) = x'^{,0} + x, \forall x \in x', x'^{,0} \in X'^{,0}$.

(ii) $(x'', x''^{,0}, x''_0)$ is additively commutative iff
$(x + x''^{,0}) = (x''^{,0} + x), \forall x \in x'', x''^{,0} \in X''^{,0}$.

(iii) $(x, x^{0,0}, x_{0,0})$ is additively commutative iff
$(x + x^{0,0}) = (x''^{,0} + x), \forall x \in x, x^{0,0} \in X^{0,0}$.

Proof. (i) To prove the condition sufficient, let e be the identity for multiplication and θ the null element for $x, y \in x'$.

$(e + e + \theta)(x + y + \theta) = e(x + y + \theta) + e(x + y + \theta) + (x + y + \theta)$ [by right para-distributive property]
$= x + y + \theta + y + y + \theta + \theta x + \theta$ [by left para-distributive property]
$= x + y + x + y + \theta x + \theta y$ (4.1)

Again, $(e + e + \theta)(x + y + \theta) = (e + e + \theta)x + (e + e + \theta)y + (e + e + \theta)\theta$ [by left para-distributive property]
$= x + x + \theta x + y + y + \theta y + e + e + \theta$ [by right para-distributive property]
$= x + x + \theta x + y + y + \theta y$ (4.2)

From (4.1) and (4.2) it follows that

$y + x + y + \theta x = x + \theta x + y + y$ [by cancellation property]

or, $\theta x + y + x + y = \theta x + x + y + y$

$\therefore y + x = x + y$

Hence, X' is additively commutative.

Necessity of the condition is obvious.

(ii) Proof is similar.

(iii) $(e + e + \theta)(x + \theta x + y) = e(x + \theta x + y) + e(x + \theta x + y) + \theta(x + \theta x + y)$ [by right para-distributive property]
$= x + \theta x + y + x + \theta x + y + \theta x + \theta x\theta + \theta y$ [by left para- distributive property (4.3)

Again, $(e + e + \theta)(x + \theta x + y) = (e + e + \theta)x + (e + e + \theta)x\theta + (e + e + \theta)y$ [by left para-distributive property]
$= x + x + \theta x + x\theta + x\theta + \theta x\theta + y + y + \theta y$ [by right para-distributive property]
$= x + x + \theta x + \theta x\theta + y + y + \theta y$ [$[\because x\theta + x\theta = 0]$] (4.4)

From (4.3) and (4.4) is follows that

$x\theta + y + x + \theta x + y + \theta x + \theta x\theta = x + \theta x + \theta x\theta + y + y$ [by cancellation property]

or, $y + x + x\theta + \theta x + y + \theta x + \theta x\theta = x + \theta x + \theta x\theta + y + y$

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or, $y + x + y + \theta x + \theta x\theta = x + \theta x + y + y + \theta x\theta$

or, $y + x + y + \theta x = x + y + y + \theta x$

$\therefore x + y = y + x$ [by cancellation property]

Hence, X is commutative.

Property 4.2. Let $(X, X^{0,0}, X_{0,0})$ be an associative pararing which is additively commutative.

Let (X', X'^{0}, X_0') and $(X'', X''^{,0}, X_0'')$ be the right and left rings respectively of X. Then the subset

(i) $(x' + x'\theta, x' \in X)$ is a basal subpararing of X';

(ii) $(x'' + \theta x'', x'' \in X)$ is a basal subpararing of X'';

(iii) $(x + \theta x\theta, x \in X)$ is a semi-basal subpararing of X,

where $(x' + x'\theta)(y' + y'\theta) = x'y' + (x'y')\theta$, $(x'' + \theta x'')(y'' + \theta y'') = (x''y'') + \theta(x''y'')$;

$(x + \theta x\theta)(y + \theta y\theta) = xy + \theta(xy)\theta$, $x'y' \in X'$; x'', $y'' \in X''$; $xy \in X$.

Proof. (iii) $(x + \theta x\theta)(y + \theta y\theta) = (x + y) + \theta x\theta + \theta y\theta$ [$\because$ is additively commutative]

$= (x + y) + \theta x\theta + \theta y\theta + \theta\theta\theta$

$= (x + y + \theta x\theta + y\theta + \theta\theta)$ [by left para-distributive property]

$= (x + y) + \theta + (x + y + \theta)\theta$ [by right para-distributive property]

$= (x + y) + \theta + (x + y)\theta$ (4.5)

By definition $(x + \theta x\theta)(y + \theta y\theta) \in X$.

Obviously, $(x + \theta x\theta)(y + \theta y\theta)(\tau + \theta + \theta) = \{(x + \theta x\theta)(y + \theta y\theta)\}(\tau + \theta + \theta)$. Now

$(x + \theta x\theta)\{(y + \theta y\theta)\} + (\tau + \theta\tau\theta) + (\omega + \theta\,\omega\theta) = (x + \theta x\theta)\{(y + \tau + \omega) + \theta(y + \tau + \omega\theta)\}$ [by (4.5)]

$= x(y + \tau + \omega) + \theta\{x(y + \tau + \omega)\theta$

$= (xy + x\tau + x\omega) + \theta\{xy + x\tau + x\omega)\theta$ [by left para-distributive property]

$= \{xy + \theta(xy)\theta\} + \{x\tau + \theta(x\tau)\theta\} + \{x\omega + \theta(x\omega)\theta$ [by (4.5)]

$= (x + \theta x\theta)\{(y + \theta y\theta)\} + (x + \theta\tau\theta)(\tau + \theta\tau\theta) + (x + \theta x\theta) + (\omega + \theta\omega\theta)$

Hence, left para-distributive property holds. Similarly, right para-distribstive property can also be verified.

So, $(x + \theta x\theta, x \in X)$ is a subpararing of X. Again,

$\theta(x + \theta x\theta)\theta = \{\theta(x + \theta x\theta)\}\theta$

$= \{\theta(x + \theta x\theta) + \theta\}\theta$

$= (\theta x + \theta x\theta + \theta)\theta$ [by left para-distibutive and associative properties]

$= \theta x\theta + \theta x\theta + \theta$ [by right para-distributive and associative properties]

$= \theta$ [$\because X^{0,0}$ is periodic of order 2]

Hence, $(x + \theta x\theta, x \in X)$ is a semi-basal sıubpararing of X

(i) Property (ii) can be similarly verified.

(i) Property (i) can be verified similarly.

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