EQUILIBRIUM ANALYSIS OF THE SEIR MATHEMATICAL MODEL

[1]Pappu Mahto,[2]Smita Dey

[1]Assistant Professor,[2]Associate Professor,
[1] Department of Mathematics,
[1] St. Xavier's College, Ranchi, India.

Abstract: Here, we present a compartmental (Susceptible – Exposed – Infected - Recovered) mathematical model. We discuss an epidemic depends on the value of the Basic Reproduction Number ($\mathcal{R}_0$). Mathematical models have been used to understand the dynamics of the disease. They have also been used as health policy tools to predict the effect of public health interventions on mitigating future epidemics or pandemics. In this study, we find out the effect of the introduction of some treatment at Exposed compartment in SEIR epidemiological model to study the effect and to control the disease and also the stability analysis. There are two types of equilibrium points. First is the disease – free equilibrium point and second is endemic equilibrium point. The disease – free equilibrium (DFE) point is stable when $\mathcal{R}_0 \leq 1$, and the endemic equilibrium (EE) point is stable when $\mathcal{R}_0 > 1$.

***Key Words* - SEIR Model, Basic Reproduction Number, Stability analysis.**

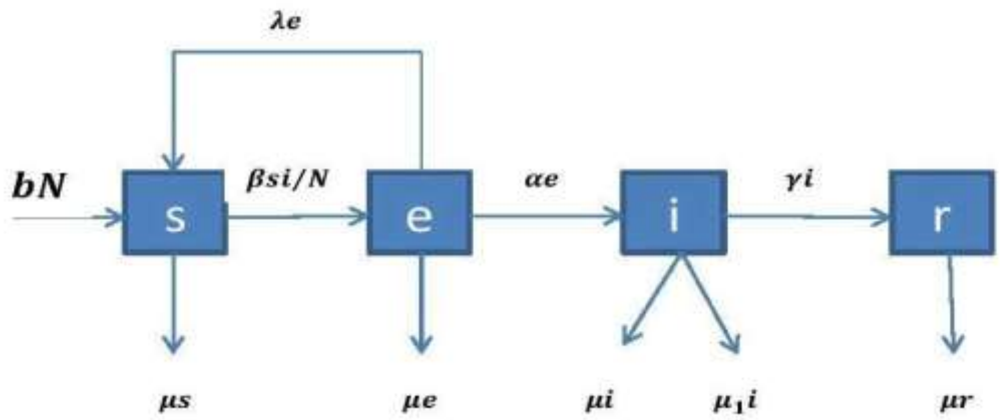
I. Introduction

In eighteen century the study of epidemic modeling starts [1].The diseases spread due to virus or bacteria that somehow enter into the body and make the individual ill, which affect the development of the community and country. It motivates to study the illness. For controlling and eliminating the disease, we have to study several stages such as spreading of the diseases, treatment and effect of vaccines, etc. The first mathematical model that could be used to describe a disease was developed in the 20th century by Kermack and McKendric [2]. The model is known as the Susceptible – Infectious – Recovered (SIR) model. The severity of the epidemic depends upon the value of the basic reproduction number ($\mathcal{R}_0$). $\mathcal{R}_0$ is defined as the average number of infections that one infective generates, in an entire susceptible population, during the time they are infectious. If $\mathcal{R}_0$ > 1, an epidemic will occur and if $\mathcal{R}_0$ < 1, the outbreak will die out.

Mathematical models are the useful tool for analyzing and checking the different hypothesis and about the spreading pattern of the diseases and then to provide useful control and prevention measures. The basic procedure in modeling the spreading of disease by using the compartmental model. There are some studies in which the mathematical model is used on epidemic diseases [3, 4, 5, 6]. Some study on SEIR model is also done previously [7, 8, 9, 10, 11].

II. THE MATHEMATICAL MODEL

The SEIR model is developed by divided into s, e, i and r compartments. The following diagram shows the transmission dynamics -



The notations we use are –

s (t) – susceptible populations,

e (t) – exposed populations,

i (t) - infected populations

r (t) – recovered populations

β – rate with which susceptible populations goes to the exposed compartment,

α – rate with which exposed population goes to the infected compartment,

γ – rate with which the infected populations goes to recovered compartment,

λ – rate with which exposed population gets some treatment and again becomes susceptible,

b – birth and immigration rate,

μ – leaving rate (natural death and emigrants rate),

μ_1 – rate with which infected population dies due to disease.

The Mathematical Model –

Mathematically, it can be written as the system of first order ODE as follows –

$$\frac{ds}{dt} = b\mathrm{N} - \mu \mathrm{s} - \frac{\beta \mathrm{si}}{N} + \lambda e$$

$$\frac{de}{dt} = \frac{\beta \mathrm{si}}{N} - (\lambda + \mu + \alpha)\, \mathrm{e}$$

$$\frac{di}{dt} = \alpha \mathrm{e} - (\mu_1 + \mu + \gamma)\, \mathrm{i}$$

$$\frac{dr}{dt} = \gamma \mathrm{i} - \mu \mathrm{r}$$

Now, putting NS for s , NE for e, NI for i and NR for r, where S, E, I and R are the proportion of susceptible, exposed, infectious and recovered proportion given by the following equations -

$$\frac{dS}{dt} = b - \mu \mathrm{S} - \beta \mathrm{SI} + \lambda E$$

$$\frac{dE}{dt} = \beta \mathrm{SI} - (\lambda + \mu + \alpha)\, \mathrm{E} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{dI}{dt} = \alpha \mathrm{E} - (\mu_1 + \mu + \gamma)\, \mathrm{I}$$

$$\frac{dR}{dt} = \gamma \mathrm{I} - \mu \mathrm{R}$$

Then, $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$

$=(b - \mu \mathrm{S} - \beta \mathrm{SI} + \lambda E) + (\beta \mathrm{SI} - (\lambda + \mu + \alpha)\, \mathrm{E}) + (\alpha \mathrm{E} - (\mu_1 + \mu + \gamma)\, \mathrm{I}) + (\gamma \mathrm{I} - \mu \mathrm{R})$

$= b - \mu(\mathrm{S} + \mathrm{E} + \mathrm{I} + \mathrm{R}) - \mu_1 I$

$= b - \mu N - \mu_1 I$ where N = S + E + I + R

So, the system has variable population size.

III. Equilibrium point –

Here we obtain DFE and EE Points. For this, we equate the system of equations to zero i.e. $\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$.

i.e. $b - \mu S - \beta SI + \lambda E = 0$……......(2)

$\beta SI - (\lambda + \mu + \alpha) E = 0$…………(3)

$\alpha E - (\mu_1 + \mu + \gamma) I = 0$…………..(4)

$\gamma I - \mu R = 0$……………………......(5)

IV. DFE point –

From equation (2), $b - \mu S - \beta SI + \lambda E = 0$

$\Rightarrow b + \lambda E = S(\beta I + \mu) \Rightarrow S = \frac{b + \lambda E}{\beta I + \mu}$

But at the disease – free state,

$\beta = 0 \Rightarrow S = \frac{b + \lambda E}{\mu}$

From equation (3), $\beta SI = (\lambda + \mu + \alpha) E = 0 \Rightarrow E = \frac{\beta SI}{(\lambda+\mu+\alpha)}$

Since $\beta = 0 \Rightarrow E = 0$.

From equation (4), $\alpha E = (\mu_1 + \mu + \gamma) I \Rightarrow I = \frac{\alpha E}{(\mu_1+\mu+\gamma)}$

Since, $E = 0 \Rightarrow I = 0$.

From equation (5), $\gamma I = \mu R \Rightarrow R = \frac{\gamma I}{\mu}$

Since, $I = 0 \Rightarrow R = 0$.

Thus, we have the DFE Point is $E_0 = \left(\frac{b + \lambda E}{\mu}, 0,0,0\right)$,

but here E = 0 So, $E_0 = (S', E', I', R') = \left(\frac{b}{\mu}, 0,0,0\right)$.

V. EE point –

It indicates that the disease or infection will persist in the system. Let EE is $E^{**}(S^*, E^*, I^*, R^*)$.

Consider equations (3) and (4), we get,

$\beta SI - (\lambda + \mu + \alpha) E = 0 \Rightarrow \beta SI = (\lambda + \mu + \alpha) E$

And $\alpha E - (\mu_1 + \mu + \gamma) I = 0 \Rightarrow (\mu_1 + \mu + \gamma) I = \alpha E$

Dividing both the terms, we get, $\frac{\beta S}{(\mu_1+\mu+\gamma)} = \frac{(\lambda+\mu+\alpha)}{\alpha} \Rightarrow S = \frac{(\lambda+\mu+\alpha)(\mu_1+\mu+\gamma)}{\alpha\beta} > 0$.

Similarly, adding equation (2) and (3), we get,

$b - \mu S - \beta SI + \lambda E + \beta SI - (\lambda + \mu + \alpha) E = 0$

$\Rightarrow b - \mu S - (\mu + \alpha) E = 0$

$\Rightarrow S = \frac{b - (\mu+\alpha) E}{\mu}$ ………………………………………………………. (6)

Again from equation (4), αE = $(\mu_1 + \mu + \gamma)$ I ⇒ I = $\frac{\alpha E}{(\mu_1+\mu+\gamma)}$………….. (7)

From equation (3), we have, βSI = $(\lambda + \mu + \alpha)$ E …………….. (8)

From equation (6) and (7), substituting the value of S & I in equation (8), we get,

$$(\lambda + \mu + \alpha) E = \beta \left\{\frac{b-(\mu+\alpha) E}{\mu}\right\}\left\{\frac{\alpha E}{(\mu_1+\mu+\gamma)}\right\} = \beta \left\{\frac{b\alpha E - \alpha(\mu+\alpha) E^2}{\mu(\mu_1+\mu+\gamma)}\right\}$$

$$\Rightarrow E\left[\frac{-\alpha\beta(\mu+\alpha) E}{\mu(\mu_1+\mu+\gamma)} + \frac{b\alpha\beta}{\mu(\mu_1+\mu+\gamma)} - (\lambda + \mu + \alpha)\right] = 0$$

So, either E = 0 Or, $\frac{-\alpha\beta(\mu+\alpha) E}{\mu(\mu_1+\mu+\gamma)} + \frac{b\alpha\beta}{\mu(\mu_1+\mu+\gamma)} - (\lambda + \mu + \alpha)$

$$\Rightarrow \frac{-\alpha\beta(\mu+\alpha) E}{\mu(\mu_1+\mu+\gamma)} = (\lambda + \mu + \alpha) - \frac{b\alpha\beta}{\mu(\mu_1+\mu+\gamma)}$$

$$\Rightarrow E = \frac{b}{(\mu+\alpha)} - \frac{\mu(\mu_1+\mu+\gamma)(\lambda+\mu+\alpha)}{\alpha\beta(\mu+\alpha)}$$

Therefore, $E^* = \frac{b}{(\mu+\alpha)} - \frac{\mu(\mu_1+\mu+\gamma)(\lambda+\mu+\alpha)}{\alpha\beta(\mu+\alpha)} = \frac{b}{(\mu+\alpha)}\left[1 - \frac{\mu(\mu_1+\mu+\gamma)(\lambda+\mu+\alpha)}{\alpha\beta b}\right]$

Let, $R_0 = \frac{\alpha\beta b}{\mu(\mu_1+\mu+\gamma)(\lambda+\mu+\alpha)}$

So, we have, $E^* = \frac{b}{(\mu+\alpha)}\left[1 - \frac{1}{R_0}\right]$ …………………..………………. (9)

Also, S = $S^* = \frac{(\lambda+\mu+\alpha)(\mu+\gamma)}{\alpha\beta} = \frac{b}{\mu R_0}$

Now consider I, from equation (7), I = $\frac{\alpha E}{(\mu_1+\mu+\gamma)}$

Substituting E^* for E, we get,

$I^* = \frac{\alpha b}{(\mu_1+\mu+\gamma)(\mu+\alpha)}\left[1 - \frac{1}{R_0}\right]$…………………………………………..(10)

And from equation (5), we get,

γI = μR ⇒ R = $\frac{\gamma I}{\mu}$

Substituting the value of I^* for I in above, we get, $R^* = \frac{\gamma\alpha b}{\mu(\mu_1+\mu+\gamma)(\mu+\alpha)}\left[1 - \frac{1}{R_0}\right]$ ………………..(11)

So, we have, S^*, E^*, I^* and R^* all are positive, $E^{**}(S^*, E^*, I^*, R^*) > 0$ if $1 - \frac{1}{R_0} > 0 \Rightarrow 1 > \frac{1}{R_0} \Rightarrow R_0 > 1$.

Here, E^{**} represents an endemic state.

VI. Stability of the equilibrium points –

Let us suppose that, k = $b - \mu S - \beta SI + \lambda E$

$l = \beta SI - (\lambda + \mu + \alpha) E$

m = $\alpha E - (\mu_1 + \mu + \gamma) I$

n = $\gamma I - \mu R$

Then, the Jacobian matrix is,

$$J = \begin{bmatrix} \frac{\partial k}{\partial S} & \frac{\partial k}{\partial E} & \frac{\partial k}{\partial I} & \frac{\partial k}{\partial R} \\ \frac{\partial l}{\partial S} & \frac{\partial l}{\partial E} & \frac{\partial l}{\partial I} & \frac{\partial l}{\partial R} \\ \frac{\partial m}{\partial S} & \frac{\partial m}{\partial E} & \frac{\partial m}{\partial I} & \frac{\partial m}{\partial R} \\ \frac{\partial n}{\partial S} & \frac{\partial n}{\partial E} & \frac{\partial n}{\partial I} & \frac{\partial n}{\partial R} \end{bmatrix} = \begin{bmatrix} -\beta I - \mu & \lambda & -\beta S & 0 \\ \beta I & -(\lambda + \mu + \alpha) & \beta S & 0 \\ 0 & \alpha & -(\mu_1 + \mu + \gamma) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \quad \text{…..(12)}$$

(i) Disease - free equilibrium point and its Stability –

For this we have, $S = \frac{b}{\mu}$, I = 0, then we get,

$$J(E_0) = \begin{bmatrix} -\mu & \lambda & -\beta\frac{b}{\mu} & 0 \\ 0 & -(\lambda+\mu+\alpha) & \beta\frac{b}{\mu} & 0 \\ 0 & \alpha & -(\mu_1+\mu+\gamma) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix} \quad \ldots\ldots\ldots\ldots(13)$$

Then the characteristic equation is – det. $[J(E_0) - \lambda' I_4] = 0$

$$\Rightarrow \begin{vmatrix} -\mu-\lambda' & \lambda & -\beta\frac{b}{\mu} & 0 \\ 0 & -(\lambda+\mu+\alpha)-\lambda' & \beta\frac{b}{\mu} & 0 \\ 0 & \alpha & -(\mu_1+\mu+\gamma)-\lambda' & 0 \\ 0 & 0 & \gamma & -\mu-\lambda' \end{vmatrix} = 0$$

$$\Rightarrow (-\mu-\lambda') \begin{vmatrix} -(\lambda+\mu+\alpha)-\lambda' & \beta\frac{b}{\mu} & 0 \\ \alpha & -(\mu_1+\mu+\gamma)-\lambda' & 0 \\ 0 & \gamma & -\mu-\lambda' \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\lambda')^2\,[(\lambda+\mu+\alpha+\lambda')(\mu_1+\mu+\gamma+\lambda') - \alpha\beta\frac{b}{\mu}] = 0$$

Either, $(\mu+\lambda')^2 = 0 \Rightarrow \lambda'_1 = -\mu\,,\ \lambda'_2 = -\mu$

Or, $(\lambda+\mu+\alpha+\lambda')(\mu_1+\mu+\gamma+\lambda') - \alpha\beta\frac{b}{\mu} = 0$

$$\Rightarrow \mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu_1\lambda'+\mu\lambda+\mu^2+\alpha\mu+\mu\lambda'+\lambda\gamma+\mu\gamma+\alpha\gamma+\gamma\lambda'+\lambda\lambda'+\mu\lambda'+\alpha\lambda'+\lambda'^2-\alpha\beta\frac{b}{\mu} = 0$$

$$\Rightarrow \lambda'^2+\lambda'(\mu_1+2\mu+\gamma+\lambda+\alpha)+(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu\lambda+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta\frac{b}{\mu}) = 0$$

Let, $\mu_1+2\mu+\gamma+\lambda+\alpha = a_1$

$\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu\lambda+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta\frac{b}{\mu} = a_2$

Then the above equation becomes $\lambda'^2 + a_1\lambda' + a_2 = 0$.

If it has two negative real roots, then det. $[J(E_0) - \lambda' I_4] = 0$, has all four eigenvalues are negative, then the DFE point is locally asymptotically stable (by stability criterion).

(ii) EE point and its Stability–

The point is $E^{**}(S^*, E^*, I^*, R^*)$ as $S^* = \frac{b}{\mu R_0}$, $E^* = \frac{b}{(\mu+\alpha)}\left[1-\frac{1}{R_0}\right]$, $I^* = \frac{\alpha b}{(\mu_1+\mu+\gamma)(\mu+\alpha)}\left[1-\frac{1}{R_0}\right]$,

$R^* = \frac{\gamma\alpha b}{\mu(\mu_1+\mu+\gamma)(\mu+\alpha)}\left[1-\frac{1}{R_0}\right]$. Here, all the points of E^{**} all are positive, if $1-\frac{1}{R_0} > 0 \Rightarrow 1 > \frac{1}{R_0} \Rightarrow R_0 > 1$.

Then, E^{**} represents an endemic points.

Then, the Jacobian of endemic equilibrium point is –

$$J(E^{**}) = \begin{bmatrix} -\beta I^*-\mu & \lambda & -\beta S^* & 0 \\ \beta I^* & -(\lambda+\mu+\alpha) & \beta S^* & 0 \\ 0 & \alpha & -(\mu_1+\mu+\gamma) & 0 \\ 0 & 0 & \gamma & -\mu \end{bmatrix}$$

Since, det. $[J(E^{**}) - \lambda' I_4] = \begin{bmatrix} -\beta I^*-\mu-\lambda' & \lambda & -\beta S^* & 0 \\ \beta I^* & -(\lambda+\mu+\alpha)-\lambda' & \beta S^* & 0 \\ 0 & \alpha & -(\mu_1+\mu+\gamma)-\lambda' & 0 \\ 0 & 0 & \gamma & -\mu-\lambda' \end{bmatrix} = 0$

$$\Rightarrow (-\mu-\lambda')\begin{vmatrix} -\beta I^*-\mu-\lambda' & \lambda & -\beta S^* \\ \beta I^* & -(\lambda+\mu+\alpha)-\lambda' & \beta S^* \\ 0 & \alpha & -(\mu_1+\mu+\gamma)-\lambda' \end{vmatrix} = 0$$

$\Rightarrow (-\mu-\lambda')\ [(-\beta I^*-\mu-\lambda')\{\ (\lambda+\mu+\alpha+\lambda')(\mu_1+\mu+\gamma+\lambda') - \alpha\beta S^*\}-\beta I^*\{\ \lambda(-\mu_1-\mu-\gamma-\lambda')+\alpha\beta S^*\}] = 0$

Either $-\mu-\lambda' = 0 \Rightarrow \lambda' = -\mu$

Or, $[(-\beta I^*-\mu-\lambda')\ \{\ (\lambda+\mu+\alpha+\lambda')(\mu_1+\mu+\gamma+\lambda') - \alpha\beta S^*\}-\beta I^*\{\ \lambda(-\mu_1-\mu-\gamma-\lambda')+\alpha\beta S^*\} = 0$

$\Rightarrow(-\beta I^*-\mu-\lambda')(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu_1\lambda'+\lambda\mu+\mu^2+\alpha\mu+\lambda'\mu+\lambda\gamma+\mu\gamma+\alpha\gamma+\gamma\lambda'+\lambda\lambda'+\mu\lambda'+\alpha\lambda'+\lambda'^2-\alpha\beta S^*)-\beta I^*(-\mu_1\lambda-\mu\lambda-\gamma\lambda-\lambda\lambda'+\alpha\beta S^*) = 0$

$\Rightarrow -\beta I^*(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu_1\lambda'-\mu_1\lambda+\mu^2+\alpha\mu+\lambda'\mu+\mu\gamma+\alpha\gamma+\gamma\lambda'+\mu\lambda'+\alpha\lambda'+\lambda'^2)+(-\mu-\lambda')(\ \mu_1\lambda+\mu_1\mu+\mu_1\alpha+\mu_1\lambda'+\lambda\mu+\mu^2+\alpha\mu+\lambda'\mu+\lambda\gamma+\mu\gamma+\alpha\gamma+\gamma\lambda'+\lambda\lambda'+\mu\lambda'+\alpha\lambda'+\lambda'^2-\alpha\beta S^*) = 0$

$\Rightarrow\quad -\beta I^*(\mu_1\mu+\mu_1\alpha+\mu^2+\alpha\mu+\mu\gamma+\alpha\gamma)-\lambda'\beta I^*(\mu_1+2\mu+\gamma+\alpha)-\beta I^*\lambda'^2-(\mu+\lambda')[(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*)+\lambda'(\mu_1+2\mu+\gamma+\lambda+\alpha)+\lambda'^2] = 0$

$\Rightarrow -\beta I^*(\mu_1\mu+\mu_1\alpha+\mu^2+\alpha\mu+\mu\gamma+\alpha\gamma)-\lambda'\beta I^*(\mu_1+2\mu+\gamma+\alpha)-\beta I^*\lambda'^2-\mu(\ \mu_1\lambda+\mu_1\mu+\mu_1\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*)-\mu\lambda'(\mu_1+2\mu+\gamma+\lambda+\alpha)-\mu\lambda'^2-\lambda'(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*)-\lambda'^2(\mu_1+2\mu+\gamma+\lambda+\alpha)-\lambda'^3 = 0$

$\Rightarrow \lambda'^3+\lambda'^2(\mu_1+2\mu+\gamma+\lambda+\alpha+\mu+\beta I^*)+\lambda'(2\beta I^*\mu+\beta I^*\gamma+\beta I^*\alpha+\beta I^*\mu_1+2\mu^2+\mu_1\mu+\mu\gamma+\mu\lambda+\mu\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*+\mu_1\lambda+\mu_1\mu+\mu_1\alpha)+[\beta I^*(\mu_1\mu+\mu_1\alpha+\mu^2+\alpha\mu+\mu\gamma+\alpha\gamma)+\mu(\mu_1\lambda+\mu_1\mu+\mu_1\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*)] = 0$

Let A = $(\mu_1+2\mu+\gamma+\lambda+\alpha+\mu+\beta I^*) = (\mu_1+3\mu+\gamma+\lambda+\alpha+\beta I^*)$

B = $(2\beta I^*\mu+\beta I^*\gamma+\beta I^*\alpha+\beta I^*\mu_1+2\mu^2+\mu_1\mu+\mu\gamma+\mu\lambda+\mu\alpha+\lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*+\mu_1\lambda+\mu_1\mu+\mu_1\alpha)$

$=(2\beta I^*\mu+\beta I^*\gamma+\beta I^*\alpha+\beta I^*\mu_1+3\mu^2+\mu_1\mu+2\mu\gamma+2\mu\lambda+2\mu\alpha+\lambda\gamma+\alpha\gamma-\alpha\beta S^*+\mu_1\lambda+\mu_1\mu+\mu_1\alpha)$

C = $\beta I^*(\mu_1\mu+\mu_1\alpha+\mu^2+\alpha\mu+\mu\gamma+\alpha\gamma)+\mu(\ \lambda\mu+\mu^2+\alpha\mu+\lambda\gamma+\mu\gamma+\alpha\gamma-\alpha\beta S^*+\mu_1\lambda+\mu_1\mu+\mu_1\alpha)$

Then, det. [J (E^{**}) $-\lambda' I_4$] becomes $\lambda'^3+A\lambda'^2+B\lambda'+C=0.$

If A and C are positive and AB – C > 0 holds then det. [J (E^{**}) $-\lambda' I_4$] = 0 has all real part are negative and thus E^{**} is stable (by Routh – Hurwitz Stability Criterion). Hence, E^{**} is stable if $R_0 > 1$.

VII. Conclusion –

SEIR mathematical model where treatment at the exposed population has been formulated and analyzed over the entire population. The model was developed based on the assumption of variable population size. The model shows that the treatment at exposed population is very important in controlling and eliminating the disease from the system. Thus, if more and more people from the population at latent period go for treatment at the exposed compartment, disease will be reduced with time. The DFE point is stable when $R_0 \leq 1$, and the EE point is stable when $R_0 > 1$. Thus, for DFE point when stable, disease disappears from the system and for EE point when stable, disease will present in the system.

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