# ON LAGRANGE-HERMITE INTERPOLATING POLYNOMIAL

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***Abstract: In this paper, we consider the Lagrange-Hermite Interpolation on uniformly distributed zeros of the unit circle with its derivative at $\pm 1$. We obtain the explicit representation of the interpolatory polynomial, and establish a convergence theorem for the same.***
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## 1 INTRODUCTION

In a paper, J.F.Traub $[6]$ considered Lagrange-Hermite interpolation process for rational function .After that, several mathematicians have considered Lagrange and Hermite interpolation on a different set of nodes. In 2012, G. Mastroianni, G. V. Milovanovic and I. Notarangelo $[4]$ considered the Lagrange-Hermite interpolation at the Jacobi zeros with its first $(r-1)$ derivatives at $\pm 1$. After that G. Mastroianni, I. Notarangelo and P. Pastore $[5]$ considered the Lagrange-Hermite interpolating polynomial at some orthogonal polynomial with its first $(r-1)$ derivatives at $0$.

In 2011. Author $[1]$ considered $(0;0,1)$ interpolation on two disjoint sets of nodes on the unit circle. After that author ( M. Shukla) $[2]$ considered the convergence of Lagrange-Hermite interpolation on the nodes, which are obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $(1-x^2)P_n^{(\alpha,\beta)'}(x)$ onto the unit circle. Recently author (with Varun ) $[3]$ considered the convergence of the Lagrange-Hermite interpolation on the non-uniformly distributed zeros on the unit circle with its first derivative at $\pm 1$.

In the present paper, we consider the Lagrange-Hermite interpolation on the uniformly distributed zeros on the unit circle, in which the Lagrange data are prescribed at all points, whereas the first derivative at $\pm 1$. In section , we describe the problem and its existence, in section 3 & 4, we give the explicit forms and estimates of the interpolatory polynomials and lastly in section 5, convergence theorem is given.

## 2 THE PROBLEM AND EXISTENCE

Let $\{z_k\}_{k=1}^{2n}$ be the zeros of

$$W(z) = z^{2n} - 1 = \prod_{k=1}^{2n}(z - z_k) \tag{2.1}$$

on the unit circle such that $z_1 = 1$ and $z_{2n} = -1$. Here we are interested to determine the convergence of interpolatory polynomials $R_n(z)$ of degree at most $2n+1$ satisfying the conditions

$$\left.\begin{aligned} R_n(z_k) &= \alpha_k ; k = 1(1)2n \\ R_n'(z_k) &= \beta_k ; k = 1,.2n \end{aligned}\right\} \tag{2.2}$$

where $\alpha_k' s$ and $\beta_k' s$ are arbitrary complex numbers. We call it as Lagrange-Hermite interpolating polynomial on the unit circle.

The fundamental polynomial of Lagrange interpolation based on the zeros of $W(z)$ is given by

$$l_k(z) = \frac{W(z)}{(z - z_k)W'(z_k)}, k = 1(1)2n \tag{2.3}$$

**Theorem 1** *There exists an interpolatory polynomial* $R_n(z)$ *on the unit circle satisfying* $(2.2)$.

**Proof.** Let $R_n(z) = W(z)q(z)$, where $q(z)$ is a linear polynomial.

Obviously $R_n(z_k)=0$ for $k=1(1)2n.$

From $R_n^{'}(\pm 1)=0,$ we have $q(z)\equiv 0.$

Hence the theorem follows.

## 3 EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall take $R_n(z)$ satisfying the conditions $(2.2)$ be given as

$$R_n(z)=\sum_{k=1}^{2n}\alpha_k A_k(z)+\sum_{k=1,2n}\beta_k B_k(z) \tag{3.1}$$

where $A_k(z)$ and $B_k(z)$ are fundamental polynomials of first and second kind respectively each of degree at most $2n+1$ satisfying the conditions:

For $k=1(1)2n$

$$\left.\begin{aligned} A_k(z_j)&=\delta_{kj};\ j=1(1)2n \\ A_k^{'}(z_j)&=0;\ j=1,2n \end{aligned}\right\} \tag{3.2}$$

For $k=1,2n$

$$\left.\begin{aligned} B_k(z_j)&=0;\ j=1(1)2n \\ B_k^{'}(z_j)&=\delta_{kj};\ j=1,2n \end{aligned}\right\} \tag{3.3}$$

**Theorem 2** *For* $k=2(2)2n-1.$ *we have*

$$A_k(z)=l_k(z)+\frac{(z+z_k)W(z)}{(z_k^2-1)W^{'}(z_k)} \tag{3.4}$$

and for $k=1,2n$

$$A_k(z)=(z+z_k)l_k(z)\left\{\frac{1}{2z_k}-\left(n-\frac{1}{4}\right)\frac{1}{z_k^2}(z-z_k)\right\} \tag{3.5}$$

$$B_k(z)=\frac{(z+z_k)W(z)}{2z_kW^{'}(z_k)} \tag{3.6}$$

One can prove $(3.4)$ and $(3.5)$ owing to condition $(3.2)$ and using $(3.3),$ we get $(3.6).$

## 4 ESTIMATION OF FUNDAMENTAL POLYNOMIALS

**Lemma 1:** *Let* $l_k(z)$ *be given by* $(2.3).$ *Then we have*

$$\max_{|z|=1}\sum_{k=1}^{2n}|l_k(z)|_{,,}\leq c\log n \tag{4.1}$$

where $c$ is a constant independent of $n$ and $z.$

**Lemma 2:** *Let* $A_k(z)$ *be given by* $(3.4)-(3.5).$ *Then we have*

$$\max_{|z|=1}\sum_{k=1}^{2n}|A_k(z)|_{,,}\leq c\log n \tag{4.2}$$

where $c$ is a constant independent of $n$ and $z$.

**Proof.** Let $z = x + iy$, and $|z| = 1$, using Lemma 1 and the inequality

$$\left(1 - x_k^2\right) \geq \left(\frac{k}{n}\right)^2 \tag{4.3}$$

where $x_k \in (-1,1)$. The lemma follows.

**Lemma 3:** *Let $B_k(z)$ be given by $(3.6)$. Then we have*

$$\max_{|z|=1} |B_k(z)|,\leq \frac{c}{n}, k = 1,2n \tag{4.4}$$

where $c$ is a constant independent of $n$ and $z$.

**Proof.** Proof of the lemma is similar to lemma 2.

## 5 CONVERGENCE

In this section , we prove the following:

**Theorem 3** *Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then sequence $\{R_n\}$ defined by*

$$R_n(z) = \sum_{k=1}^{2n} f(z_k) A_k(z) + \sum_{k=1,2n} \beta_k B_k(z) \tag{5.1}$$

*converges uniformly to $f(z)$.*

To prove the theorem 3 , we shall need the following:

**Jackson' Inequality:** Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then there exists a polynomial $F_n(z)$ of degree at most $2n+1$ such that

$$F_n(z) = \sum_{k=1}^{2n} F_n(z_k) A_k(z) + \sum_{k=1,2n} F_n'(z_k) B_k(z)$$

satisfying the conditions

$$\left|F_n(z) - f(z)\right| \leq c\omega\left(f, \frac{1}{n}\right) \tag{5.2}$$

**O. Kiš Inequality:**

$$\left|F_n^{(m)}(z)\right| \leq cn^m \omega\left(f, \frac{1}{n}\right), m \in I^+ \tag{5.3}$$

**Proof.** Let $z = \exp i\theta$, $(0 \leq \theta < 2\pi)$ and arbitrary numbers $\beta_k'$ s be such that

$$\left|\beta_k\right| = O\left(n\omega\left(f, \frac{1}{n}\right)\right), k = 1,2n \tag{5.4}$$

Using Lemmas 2 & 3 and $(5.1)-(5.4)$, The theorem follows.

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