EXPLICIT CONSTRUCTIONS FOR FAMILIES OF EXPANDERS

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Abstract : Expander graphs are higly connected sparse finite graphs, which play an important role in Pure (number theory, group theory, geometry, and many more) and Applied Mathematics and also very important in computer science as basic building blocks for network constructions, error connecting codes, algorithms and more. While k-regular random graphs are, in fact, very good expanders, many applications require explicit, deterministic constructions of expanders. In this paper our aim is to give explicit constructions for families of expanders.

Keywords : explicit construction, family of expanders, adjacency matrix, eigenvalues, Ramanujan graph.

INTRODUCTION

Expander graphs are higly connected sparse finite graphs, which play an important role in Pure (number theory, group theory, geometry, and many more) and Applied Mathematics. They are also very important in computer science as basic building blocks for network constructions, error connecting codes, algorithms and more.

Originally introduced in the context of building robust, high-performance communication networks [2], expanders are both very natural from a purely mathematical perspective and play a key role in a host of other applications (from complexity theory to coding). While k-regular random graphs are, in fact, very good expanders [4, 7], many applications require explicit, deterministic constructions of expanders. Consequently, a rich body of literature in graph theory deals with deterministic constructions of expanders, of which the best known examples are Margulis's construction [13] (with Gabber and Galil's analysis [8]), algebraic constructions involving Cayley graphs such as that of Lubotzky, Phillips, and Sarnak [11], constructions that utilize the zig-zag product [16], and constructions that rely on the concept of 2-lifts [3, 12].

Our aim is to give explicit constructions for families of expanders.

An expander graph is a sparse graph that has strong connectivity properties, quantified using vertex, edge or spectral expansion. Intuitively, an expander is a finite, undirected multigraph in which every subset of the vertices that is not "too large" has a "large" boundary. For instance, a graph for which any "small" subset of vertices has a relatively "large" neighborhood. Conceivably, that means that removing random edges (local connection failures) does not reduce the property of an expander by much. Hence, a disconnected graph is not an expander, since the boundary of a connected component is empty.

Every connected graph is an expander. However, different connected graphs have different expansion parameters. For example, the complete graph has the best expansion property, but it has the largest possible degree. So informally, a graph is a good expander if it has low degree and high expansion parameters. A very interesting prospective of expander graphs, which also shows their importance, is that they can be considered into the human brain : Viewing neurons as vertices and axons as edges, the brain is 'as a rough first approximation' also a graph, and this graph actually appears as one of the first motivations that led to expander graphs.

Main Problem

A family of expanders is a family of graphs $G = (V, E)$, such that each graph is k-regular. An explicit construction of a family of expanders is a construction in which G is "efficiently computable" given A. The weakest sense in which a construction is said to be explicit is when, given A, the (adjacency matrix of the) graph G can be constructed in time polynomial in n. A stronger requirement, which is necessary for several applications, is that given A and $i \in \{1, 2, \ldots, A\}$, the list of neighbors of the i-th vertex of G can be computed in time polynomial in log n.

In many explicit constructions of constant-degree expanders, the construction is extremely simple, and besides satisfying the stricter definition of "explicit" above, it is also such that the adjacency list of a vertex is given by a "closed-form formula." The analysis of such constructions, however, usually requires very sophisticated mathematical tools.

Now we give explicit constructions for families of expanders.

We shall solve this problem algebraically, by appealing to the adjacency matrix A of the graph $G = (V, E)$; it is indexed by pairs of vertices x, y of G, and A_{xy} is the number of edges between x and y.

When G has n vertices, A is an n-by-n, symmetric matrix, which completely determines G. By standard linear algebra, A has n real eigenvalues, repeated according to multiplicities, that we list in decreasing order:

$\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}$.

Therefore, to have good quality expanders, the spectral gap $k - \mu_1(G_m)$ has to be as large as possible. However, the spectral gap cannot be too large as was observed independently by Alon [1], Burger [5], Serre [17] and Grigorchuk-Zuk [9]. In fact, we have the bound implied by the following result:

Theorem 1.1.2. Let $(G_m)_{m\geq 1}$ be a family of finite, connected, k-regular graphs with $|V_m| \to +\infty$ as $m \to +\infty$. Then

$$\liminf_{m\to+\infty} \mu_1(G_m) \geq 2\sqrt{k-1}.$$

This Theorem 1.1.2 singles out an extremal property on the eigenvalues of the adjacency matrix of a k-regular graph; this motivates the definition of a Ramanujan graph.

Definition 1.1.20. (Ramanujan graphs) : A finite, connected, k-regular graph G is Ramanujan if, for every eigenvalue of other than $\pm k$ one has

$$\mu_1 \leq 2\sqrt{k-1}.$$

So, if for some $k \geq 3$ we succeed in constructing an infinite family of k-regular Ramanujan graphs, we will get a solution of our main problem (hence also of Theorem 1.1.1) which is optimal from the spectral point of view.

Theorem 1.1.3. For the following values of k, there exist infinite families of k-regular Ramanujan graphs:

- $k = p + 1$, p an odd prime (Lubotzky-Phillips-Sarnak [11]; Margulis [14]);
- $k = 3$ (Chiu [8]);
- $k = q + 1$, q a prime power (Morgenstern [15]).

Our purpose in this thesis is to describe the Ramanujan graphs of Lubotzky-Phillips-Sarnak [11] and Margulis [14]. While the description of these Ramanujan graphs is elementary, the proof that they have the desired properties is not. For example, the proofs in [11] and [14] make free use of the theory of algebraic groups, modular forms, theta correspondences, and the Riemann Hypothesis for curves over finite fields. Our aim here is to give elementary and self contained proofs of most of the properties enjoyed by these graphs. Actually, our elementary methods will not give us the full strength of the Ramanujan bound for these graphs, though they do have that property. Nevertheless, we will be able to prove that they form a family of expanders with a quite good explicit asymptotic estimate on the spectral gap. This estimate is strong enough to provide explicit solutions to two outstanding problems in graph theory that we now describe.

The adjacency matrix and its spectrum

We shall be concerned with graphs $G = (V, E)$, where V is the set of vertices, E is the set of edges. As stated in the introduction, we always assume our graphs to be undirected, and most often we will deal with finite graphs.

We let $V = \{v_1 v_2 \ldots\}$ be the set of vertices of G. Then the adjacency matrix of the graph is the matrix A indexed by pairs of vertices $v_i, v_j \in V$. That is, $A = (A_{ij})$, where A_{ij} = number of edges joining v_i to v_j.

We say that G is simple if there is at most one edge joining adjacent vertices; hence G is simple if and only if $A_{ij} \in \{0, 1\}$ for every $v_i, v_j \in V$.

Note that A completely determines G and that A is symmetric because G is undirected. Furthermore, G has no loops if and only if $A_{ii} = 0$ for every $v_j \in V$.

It should be noted that if $k \geq 2$ be an integer, then we say that the graph G is k-regular if, for every $v_i \in V : \sum_{v_j \in V} A_{ij} = k$.

If G has no loop, this amounts to saying that each vertex has exactly k neighbours.

Assume that G is a finite graph on n vertices. Then A is an n-by-n symmetric matrix; hence it has n real eigenvalues, counting multiplicities, that we may list in decreasing order:

$\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}$.

The spectrum of G is the set of eigenvalues of A. Note that μ_0 is a simple eigenvalue, or has multiplicity 1, if and only if $\mu_0 > \mu_1$.

For an arbitrary graph $G = (V, E)$, consider functions $f : V \to \mathbb{C}$ from the set of vertices of X to the complex numbers, and define

$$\ell^2(V) = \left\{ f : V \to \mathbb{C} : \sum_{v \in V} |f(v)|^2 < +\infty \right\}.$$

The space $\ell^2(E)$ is defined analogously.

Clearly, if V is finite, say $|V| = n$, then every function $f : V \to \mathbb{C}$ is in $\ell^2(V)$. We can think of each such function as a vector in $\mathbb{C}^n$ on which the adjacency matrix acts in the usual way:

$$Af = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{in} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} f(v_1) + A_{12} f(v_2) + \ldots + A_{1n} f(v_n) \\ A_{i1} f(v_1) + A_{i2} f(v_2) + \ldots + A_{in} f(v_n) \\ A_{n1} f(v_1) + A_{n2} f(v_2) + \ldots + A_{nn} f(v_n) \end{pmatrix}$$

Hence $(Af)(v_i) = \sum_{j=1}^{n} A_{ij} f(v_i)$. It is very convenient, both notationally and conceptually, to forget about the numbering of vertices and to index matrix entries of A directly by pairs of vertices. So we shall represent A by a matrix $(A_{xy})_{x,y \in v}$ and the previous formula becomes $(Af)(x) = \sum_{y \in V} A_{xy} f(y)$, for every $x \in V$.

Proposition 1.1.2. Let G be a finite k-regular graph with n vertices. Then

(i) $\mu_0 = k$;

(ii) $|\mu_i| \leq k$ for $1 \leq i \leq n-1$;

(iii) μ_0 has multiplicity 1 if and only if G is connected.

Note that a graph $G = (V, E)$ is bipartite, or bicolourable, if there exists a partition of the vertices $V = V_+ \cup V_-$ such that, for any two vertices x, y with $A_{xy} \neq 0$, $x \in V_+$ if (resp. V_-), then $y \in V_-$ (resp. V_+).

In other words, it is possible to paint the vertices with two colours, in such a way that no two adjacent vertices have the same colour. Bipartite graphs have very nice spectral properties characterized by the following:

Proposition 1.1.3. Let G be a connected, k-regular graph on n vertices. The following are equivalent:

(i) G is bipartite;

(ii) the spectrum of G is symmetric about 0;

(iii) $\mu_{n-1} = -k$.

Definition 1.1.23. (Inequalities on the spectral gap) : Let $G = (V, E)$ be a graph. For $F \subseteq V$, we define the boundary ∂F of F to be the set of edges with one extremity in F and the other in $V - F$. In other words, ∂F is the set of edges connecting F to $V - F$. Note that $\partial F = \partial(V - F)$.

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