THE STUDIES ON LINEAR STOCHASTIC AND LINEAR FRACTIONAL PROGRAMMING

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ABSTRACT

Optimizing the ratio of two linear functions subject to some constraints in which at least one of the problem data is random is called linear stochastic fractional programming problem (LSFPP). In this paper, a method combining the LSFP with branch-and-bound technique is proposed. A chance-constrained linear fractional programming problem has been considered. Various cases of randomness have been taken into account. Once a real number solution is obtained by LSFP, the branch-and-bound technique is used to obtain the integer solution. The best feasible solution among the trials is taken as the final solution. This method works well if the problem is complex and the constraints are non-linear. A machine manufacturing problem has been considered where the above models have been applied. Processing time are considered independently distributed normal variables.

KEY WORDS

Stochastic, Fractional, Cluster and Non Linear.

1. INTRODUCTION

In recent years linear stochastic fractional programming problem(LSFPP) has attracted various authors. Infect most of the systems involve parameters and variables, which are random variables due to uncertainties. Probabilistic methods are powerful in modeling such systems. Stochastic programming deals with situations where uncertainty is present in the data or model. Usually in deterministic mathematical programming the data (coefficients) are known whereas in stochastic programming the data are unknown, instead we may have a probability distribution present, such models are also sometimes called chance-constrained linear programs, as the name implies, is mathematical (i.e., linear, integer, mixed-integer, non-linear) programming but with a stochastic element present in the data. For example, the modeling of an investment portfolio so as to meet random liabilities, modeling of strategic capacity investments, power systems, that is modeling the operation of electrical power supply systems so as to meet consumers demand for electricity etc.

This chapter focuses mainly on LSFPP with mixed constraints, in which some of the data is random with at least one deterministic constraints. We assume that decision variables are deterministic. The objective of this chapter is to maximize the ratio of two linear functions subject to a set of probabilistic and deterministic inequalities and non-negative constraints on the variables. We have introduced the elements of LSFPP with mixed constraints techniques. Also we have discussed the transformations of LSFPP's into non-linear programming problems. Here we have also given a machine manufacturing problem and its solution procedure.

2. FRACTIONAL PROGRAMMING

While working in programming problems it was felt to extend the work on single objective optimization problem, where the objective function is characterized as a ratio $f(x)/g(x)$ of two function. Such a problem is called a fractional programming problem. In particular, if f and g are linear or affine functions and the constraints are also linear then it is a linear fractional programming problem. A first systematic treatment of single ratio fractional programs was attempted by the author Schaible (1978)

A linear programming model may be set up to find the best combination of cargoes to loaded into a ship in terms of maximum profit. This lead to a linear programming model, which may however not be realistic, since it disregards the fact that certain types of cargo are loaded or unleaded more slowly than others. To allow for this, the ratio of profit per trip to total trip may be maximized, subject to appropriate linear constraints. This lead to a linear fractional program with n objective function $(\alpha^T x + \alpha)/(C^T x + \beta)$.

A general fractional programming problem can be formulated as follows :

(FP) Minimize $f(x) \,/\, g(x)$

Subject to $h(x) \leq 0, x \in X_0$

where X_0 is an open set in R^n, $f : X_0 \rightarrow R, g: X_0 \rightarrow R$ and $h : X_0 \rightarrow R_m$ for $g(x) > 0$ and all $x \in X$.

Let D be the set of all those points of x which satisfy constraints, then the set D is said to be the feasible region of (FP). A feasible solution X^0 of (FP) is known as an optimal solution to the program (FP) if $\frac{f(X)}{g(X)} \geq \frac{f(X^0)}{g_i(X^0)}$ for all $X \in D$

3. LINEAR FRACTIONAL PROGRAMMING PROBLEM

In real world situation a decision maker faces the problem of optimizing ratio. For example, in a manufacturing company one may have to potimize Output/Cost, Profit/Cost etc. Optimizing the ratio of two linear functions subject to some constraints is called Linear Fractional Programming Problem (LFPP). Using the notation of [Charnes & Cooper [3] 1962) the LFPP is stated as follows :

Maximize $f(x) = (c^{'}x + \alpha)/\left(d^{'}x + \beta\right).$ ------------- (1)

Subject to $Tx \leq b, \;\; x \geq 0$

where T is an $m \times n$ matrix, b is an $m \times 1$ vector, c, d and x are $n \times 1$ vectors respectively and α, β are scalars. Let F be the set of feasible solutions of this problem. Using the transformation $y = tx$

where t is a scalar., Charnes & Cooper (1962) have replaced the problem (1) by the following two problems :

Maximize $c'y + \alpha t$ --------------------------- (2)

Subject to $Ty - bt \leq 0,\ d'y + \beta t = 1$

and $y, t > 0$

Maximize $-(c'y + \alpha t)$ ------------------------ (3)

Subject to $Ty - bt \leq 0,\ d'y + \beta t = -1$

and $y, t \geq 0$

Charnes & Cooper (1962) have sown that if the denominator , $(d'x + \beta) > 0\ \forall\, x \in F,$ then the problem (3) is infeasible whereas if $(d'x + \beta) < 0\ \forall\, x \in F,$ then the problem (2) is infeasible. (Zionts (1968) have shown that for any solution of the problem (1) which is finite $\forall\, x \in F$ the denominator $(d'x + \beta)$ for all practical cases, cannot have two different signs in the feasible space defined by $Tx < b, x > 0$. The solution of the problem (1), therefore, ultimately amounts to solving only the problem (2) or the problem (3)

4. LINEAR STOCHASTIC PROGRAMMING PROBLEM

The Chance-Constrained Stochastic programming was first studied by Charnes & Copper (1954). The mathematical formulation can be states as follows

Optimize $f(x) = c'(x)$ ---------------------------- (4)

Subject to $Pr\left(\sum_{j=1}^{n} t_{ij} x_j \leq b_i\right) \geq 1 - P_i \quad x_i \geq 0,\ i = 1$ to m.

Where $1 - P_i\ (0 < P_i < 1)$ is the least probability with which ith constraint is satisfied. Assume that the radndom variables are normally distributed with known parameters. In real life situation, there are many applications of Stochastic programming. Some of these can be seen in. [Dempster (1999)]

5. LINEAR STOCHASTIC FRACTIONAL PROGRAMMING PROBLEM (LSFPP) WITH MIXED CONSTRAINS

Using the mathematical form of LSFPP [Charles [1] and Charles & Dutta (2001) we introduce LSFPP with mixed constraints as follows

Maximize $f(x) = (c^{'}x + \alpha)/\left(d^{'}x + \beta\right)$------------------ (5)

Subject to $Pr\left(\sum_{j=1}^{n} t_{ij} x_j \leq b_i\right) \geq 1 - P_i$

$\sum_{j=1}^{n} r_{kj} x_j \leq b_{rk}$

Where $T_{m\times n} = \|t_{ij}\|,\ X_{n\times 1} = \|x_j\|, b_{m\times 1} = \|b_i\|,\ b_{rk\times 1} = \|b_{rk}\|,$

$i = 1$ to m ; $j = 1$ to n; $k = 1$ to h. α, β are scalars.

6. MACHINE MANUFACTURING PROBLEM

A company manufactures n types of machines. Further, their production requires m processes. The processing time t_{ij} are known to be independently distributed normal variables with estimated means u_{ij} and standard deviations s_{ij} . We also know the average profit for each machine. The annual average fixed cost to manufacture the machine is β. The mean capital invested is Rs. d_i per machine of type i. The annual resultant mean profit is Rs. C_i per machine of type i. Below table may provide us more details.

Data for manufacture of machines

Process	Time required per unit (hours) Machine type				Avallable time per annum (hours)
	1	2	---	N	
1	u_{11} s_{11}	u_{12} s_{12}	---	u_{1n} s_{1n}	b_1
2	u_{21} s_{21}	u_{22} s_{22}	---	u_{2n} s_{2n}	b_2
3	u_{31} s_{31}	u_{32} s_{32}	---	u_{3n} s_{3n}	b_3
m	u_{ml} s_{ml}	u_{m2} s_{m2}	---	u_{mn} s_{mn}	bm
MP	c_1	c_2		c_n	
MI	d_1	d_2		d_n	
M.P. – Mean profit			M.I. Mean investrent		

To manufacture n type of machines the resource availability is given in the following table

Process	Resources required per machines Machine Type				Avallable time per annum (units)
	1	2	---	n	
1	r_{11}	r_{12}	---	r_{1n}	b_{r1}
2	r_{21}	r_{22}	---	r_{2n}	b_{r2}
⋮	⋮	⋮	---	⋮	⋮
h	r_{h1}	r_{h2}	---	r_{hn}	b_{rh}

The objective is to determine the number of machines of each type that should be produced per annum so as to maximize profitability (ratio of net profit to capital invested).

Let x_i denotes the number of machines of type i manufactured per annum. The problem then can be formulated as follows :

Maximize $\quad R(x) = c'x/(d'x + \beta) > 0$

Subject to the constraints : $Pr\left(\sum_{j=1}^{n} t_{ij}x_j \le b_i\right) \ge 1 - P_i \quad 0 < P_i \le 1, \quad \forall\, i = 1 \text{ to } m$

$\sum_{j=1}^{n} r_{kj}x_j \le b_{rk} \quad ; \quad k = 1 \text{ to } h$

$x_j \ge 0$ and integers

7. SOLUTION PROCEDURE

Let us consider the model for 2 variables

Maximize $$f(x) = (c_1x_2 + c_2x_2)/(d_1x_1 + d_2x_2 + \beta). \text{ -------------- } (6)$$

Subject to $$u_{11}x_1 + u_{12}x_2 + Kq_i\sqrt{s_{11}^2x_1^2 + s_{12}^2x_2^2} \le b_1 \quad \text{-------------- } (7)$$

$$u_{21}x_1 + u_{22}x_2 + Kq_i\sqrt{s_{21}^2x_1^2 + s_{22}^2x_2^2} \le b_2 \quad \text{--------------- } (8)$$

$$r_{11}x_1 + r_{12}x_2 \le br_1 \quad \text{--------------- } (9)$$

$$\forall\, x_1, x_2 \ge 0$$

Using the transformation $y = tx$ we obtain,

Maximize $$f(x) = c_1y_1 + c_2y_2 \quad \text{-------------- } (10)$$

Subject to $$d_1y_1 + d_2y_2 + t\beta = 1 \quad \text{-------------- } (11)$$

$$a_{11}y_1^2 + a_{12}y_2^2 + a_{13}ty_1 + a_{14}ty_2 + a_{15}y_1y_2 - b_1^2t^2 \le 0 \quad \text{-------------- } (12)$$

$$a_{21}y_1^2 + a_{22}y_2^2 + a_{23}ty_1 + a_{24}ty_2 + a_{25}y_1y_2 - b_2^2t^2 \le 0 \quad \text{-------------- } (13)$$

$$r_{11}y_1 + r_{12}y_2 - tbr_1 \le 0 \quad \text{-------------- } (14)$$

$$y_1, y_2 \ge 0$$

$$a_{j1} = S_{j1}^2K^2q_i - u_{j1}^2, a_{j2} = S_{j2}^2K^2q_i - u_{j2}^2, a_{j3} = 2b_ju_{j1}, a_{j4} = 2b_ju_{j2}, a_{j5} = -2u_{j1}u_{j2}$$
$$j = 1, 2 \dots\dots$$

Converting objective function (10) as minimizing problem and using Kuhn Tucker conditions.

we obtain

$$L(y_1, y_2; t; \lambda) = -c_1y_1 - c_2y_2 + \lambda_1[d_1y_1 + d_2y_2 + t\beta - 1]$$
$$+\lambda_2[a_{11}y_1^2 + a_{12}y_2^2 + a_{13}ty_1 + a_{14}ty_2 + a_{15}y_1y_2 - b_1^2t^2]$$
$$+\lambda_3[a_{21}y_1^2 + a_{22}y_2^2 + a_{23}ty_1 + a_{24}ty_2 + a_{25}y_1y_2 - b_2^2t^2]$$

$$+\lambda_4[r_{11}y_1 + r_{12}y_2 - tbr_1] \quad \text{---------------} (15)$$

$$g_1 = \partial L/\partial y_1 = -c_1 + \lambda_1 d_1 + \lambda_2[2a_{11}y_1 + a_{13}t + a_{15}y_2]$$
$$+\lambda_3[2a_{21}y_1 + a_{23}t + a_{25}y_2] + \lambda_4 r_{11} \quad \text{------------------} (16)$$
$$g_2 = \partial L/\partial y_2 = -c_2 + \lambda_1 d_2 + \lambda_2[2a_{12}y_2 + a_{14}t + a_{15}y_1]$$
$$+\lambda_3[2a_{22}y_2 + a_{24}t + a_{25}y_1] + \lambda_4 r_{12} \quad \text{-------------------} (17)$$
$$g_3 = \partial L/\partial t \quad = \lambda_1\beta + \lambda_2[a_{13}y_1 + a_{14}y_2 - 2b_1^2 t]$$
$$+\lambda_3[a_{23}y_1 + a_{24}y_2 - 2b_2^2 t] - \lambda_4 br_1 \quad \text{-------------------} (18)$$
$$g_4 = \partial L/\partial\lambda_1 = d_1y_1 + d_2y_2 + \beta - 1 \quad \text{-------------------} (19)$$
$$g_5 = \partial L/\partial\lambda_2 = a_{11}y_1^2 + a_{12}y_2^2 + a_{13}ty_1 + a_{14}ty_2 + a_{15}y_1y_2 - b_1^2t^2 \text{ ------ } (20)$$
$$g_6 = \partial L/\partial\lambda_3 = a_{21}y_1^2 + a_{22}y_2^2 + a_{23}ty_1 + a_{24}ty_2 + a_{25}y_1y_2 - b_1^2t^2 \text{ ----- } (21)$$
$$g_7 = \partial L/\partial\lambda_4 = r_{11}y_1 + r_{12}y_2 - tbr_1 \quad \text{--------------------------------} (22)$$

To solve Non-linear equations from (16) to (22), Newton's Method can be applied. We begin with the form $g_j = 0$; j = 1 to 7. Let $y_1 = Y_1$, $y_2 = Y_2$, t = T, $\lambda_k = A_k$ $k = 1$ to 4 be root, and expand seven functions as a Taylors Series about the point $\left(y_1^{(i)}, y_2^{(i)}; t^{(i)}; t^{(i)}; \lambda_k^{(i)}\ k = 1 \text{ to } 4\right)$in terms of $(Y_1 - y_1)$, $(Y_2 - y_2)$; $(T - t)$; $(A_k - \lambda_k)$; k = 1 to 4}, where $\left(y_1^{(i)}, y_2^{(i)}; t^{(i)}; t^{(i)}; \lambda_k^{(i)}\ k = 1 \text{ to } 4\right)$ is a point near the root.

$\partial g_1/\partial y_1 = \lambda_2 2a_{11} + \lambda_3 2a_{21};\ \partial g_1/\partial y_2 = \lambda_2 a_{15} + \lambda_3 a_{25};\ \partial g_1/\partial t = \lambda_2 a_{13} + \lambda_{13} a_{23}$

$\partial g_1/\partial\lambda_1 = d_1;\ \partial g_1/\partial\lambda_2 = 2a_{11}y_1 + a_{13}t + a_{15}y_2$

$\partial g_1/\partial\lambda_3 = 2a_{21}y_1 + a_{23}t + a_{25}y_2;\ \partial g_1/\partial\lambda_4 = r_{11}$

$\partial g_2/\partial y_1 = \lambda_2 a_{15} + \lambda_3 a_{25};\ \partial g_2/\partial y_2 = \lambda_2 2a_{12} + \lambda_3 2a_{22};\ \partial g_2/\partial t = \lambda_2 a_{14} + \lambda_3 a_{24}$

$\partial g_2/\partial\lambda_1 = d_2;\ \partial g_2/\partial\lambda_2 = 2a_{12}y_2 + a_{14}t + a_{15}y_1$

$\partial g_2/\partial\lambda_3 = 2a_{22}y_2 + a_{24}t + a_{25}y_1;\ \partial g_2/\partial\lambda_4 = r_{12}$

$\partial g_3/\partial y_1 = \lambda_2 a_{13} + \lambda_3 a_{23};\ \partial g_3/\partial y_2 = \lambda_2 a_{14} + \lambda_3 2a_{24};$

$\partial g_3/\partial t = -\lambda_2 2b_1^2 - \lambda_3 b_2^2;\ \partial g_3/\partial\lambda_1 = \beta;$

$\partial g_3/\partial\lambda_2 = a_{13}y_1 + a_{14}y_2 - 2b_1^2 t;\ \partial g_3/\partial\lambda_3 = a_{23}y_1 + a_{24}y_2 - 2b_2^2 t; \partial g_3/\partial\lambda_4 = -br_1;$

$\partial g_4/\partial y_1 = d_1;\ \partial g_4/\partial y_2 = d_2;\ \partial g_4/\partial t = \beta;$

$\partial g_4/\partial\lambda_1 = \partial g_4/\partial\lambda_2 = \partial g_4/\partial\lambda_3 = \partial g_4/\partial\lambda_4 = 0;$

$\partial g_5/\partial y_1 = 2a_{11}y_1 + a_{13}t + a_{15}y_2;\ \partial g_5/\partial y_2 = 2a_{12}y_2 + a_{14}t + a_{15}y_1;$

$\partial g_5/\partial t = a_{13}y_1 + a_{14}y_2 - 2b_1^2 t;\ \partial g_5/\partial\lambda_1 = \partial g_5/\partial\lambda_2 = \partial g_5/\partial\lambda_3 = \partial g_5/\partial\lambda_4 = 0$;

$\partial g_6/\partial y_1 = 2a_{21}y_1 + a_{23}t + a_{25}y_2;\ \ \partial g_6/\partial y_2 = 2a_{22}y_{22} + a_{24}t + a_{24}y_1$;

$\partial g_6/\partial t = a_{23}y_1 + a_{24}y_2 - 2b_2^2 t;\ \partial g_6/\partial\lambda_1 = \partial g_6/\partial\lambda_2 = \partial g_6/\partial\lambda_3 = \partial g_6/\partial\lambda_4 = 0$;

$\partial g_7/\partial y_1 = r_{11}; \partial g_7/\partial y_2 = r_{12};\ \partial g_7/\partial t = -br_1;$

$\partial g_7/\partial\lambda_1 = \partial g_7/\partial\lambda_2 = \partial g_7/\partial\lambda_3 = \partial g_7/\partial\lambda_4 = 0$;

We obtain the following system of linear simultaneous equation where the coefficient matrix is a symmetric matrix obtained after truncation of Taylor's series.

$$\begin{pmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \partial g_1/\partial t & \partial g_1/\partial\lambda_1 & \partial g_1/\partial\lambda_2 & \partial g_1/\partial\lambda_3 & \partial g_1/\partial\lambda_4 \\ & \partial g_2/\partial y_2 & \partial g_2/\partial t & \partial g_2/\partial\lambda_1 & \partial g_2/\partial_2\lambda_2 & \partial g_2/\partial\lambda_3 & \partial g_2/\partial\lambda_4 \\ & & \partial g_3/\partial t & \partial g_3/\partial\lambda_1 & \partial g_3/\partial\lambda_2 & \partial g_3/\partial\lambda_3 & \partial g_3/\partial\lambda_4 \\ & & & \partial g_4/\partial\lambda_1 & \partial g_4/\partial\lambda_2 & \partial g_4/\partial\lambda_3 & \partial g_4/\partial\lambda_4 \\ & & & & \partial g_5/\partial\lambda_2 & \partial g_5/\partial\lambda_3 & \partial g_5/\partial\lambda_4 \\ & & & & & \partial g_6/\partial\lambda_3 & \partial g_6/\partial\lambda_4 \\ & & & & & & \partial g_7/\partial\lambda_4 \end{pmatrix} \begin{pmatrix} \Delta y_1^{(i)} \\ \Delta y_2^{(i)} \\ \Delta t^{(i)} \\ \Delta\lambda_1^{(i)} \\ \Delta\lambda_2^{(i)} \\ \Delta\lambda_3^{(i)} \\ \Delta\lambda_4^{(i)} \end{pmatrix} = - \begin{pmatrix} \partial L/\partial y_1 \\ \partial L/\partial y_2 \\ \partial L/\partial t \\ \partial L/\partial\lambda_1 \\ \partial L/\partial\lambda_2 \\ \partial L/\partial\lambda_3 \\ \partial L/\partial\lambda_4 \end{pmatrix} \ldots \text{----------------- (23)}$$

is evaluated at $\left(y_1^{(i)}, y_2^{(i)}; t^{(i)}; t^{(i)}; \lambda_k^{(i)}\ k = 1 \text{ to } 4\right)$. Solving this, we compute

$$\left(y_k^{(i+1)} = y_k^{(i)} + \Delta y_k^{(i)}, k = 1,2;\ t^{(i+1)} = t^{(i)} + \Delta t^{(i)}; \lambda_k^{(i+1)} = \lambda_k^{(i)} + \Delta\lambda_k^{(i)}\ k = 1 \text{ to } 4\right)$$

We repeat this process with i replaced by $i + 1$ until RHS close to zero. The method can be extended to more than seven simultaneous equations.

8. CONCLUSION

In this chapter we have considered a machine manufacturing problem which turns out to be Linear Stochastic Fractional Programming Problem with joint constrains. In the solution procedure we have considered Kuhn Tucker conditions in modified form i.e., the constraints (12) have been considered in equality sense. The reason is that the coefficient matrix in (23) is a sparse matrix, which may leads to multiple solutions. If the solutions thus obtained, satisfy the constraints (7) to (9), we accept it which is the optimal solution guaranteed by Kuhn Tukcer conditions. If this solution does not satisfies (7) to (9) then we have to add slack variables in (12) to (14) and the system (23) has to be modified with respect to Kuhn Tucker conditions in which case the complexity of solving the system (23) will increase. The solution procedure is illustrated in (7).

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