Relativistic Quantum Mechanics: Formulation Application

***Pramendra Ranjan Singh **Abhishek Ranjan Singh ***Ramendra Kumar Singh**

**Principal, Narayan Mahavidalaya, Goreakothi, Siwan (Bihar)*
***Research Scholar, Dept. of Physics, J.P. University, Chapra(Bihar)*
****Department of Physics, Jagdam College, Chapra (Bihar)*

ABSTRACT

Non-relativistic quantum mechanics refers to the mathematical formulation of quantum mechanics applied in the context of Galilean relativity, which quantized the equation of classical mechanics by simply replacing dynamical variables by operator. While relativistic quantum mechanics in quantum mechanics dealsin terms of special relativity. Relativistic quantum (RQM) is any Poincare' covariant formulation of quantum mechanics deals in terms of special relativity. It is valid for massless particle as wellas for massive particle for all range of velocity upto velocity of light. The theory has vast application in accelerator physics, high energy physics, particle physics, atomic physics, condensed matter physics and chemistry.

I. INTRODUCTION

Earlier, Before RQM, for getting prediction of Antimatter, spin magnetic moments of elementary spin (½) fermions, quantum dynamics of charged particles in electromagnetic fields and fine structure, terms have to be introduced artificially into the Hamiltonion operator to achieve agreement with the experimental observations facts.The relativistic quantum field theory is most successful because even conservation of particle number during matter creation and annihilation is absolutely justified by this theory. Here, we try to correlate special theory of relativity with quantum mechanics to make relativistic quantum mechanics to more applicable and wider implacable.

II. Combination of special relativity and quantum mechanics

As postulate of quantum mechanics the time dependent Schrödinger equation is;

$$i \quad \hbar \frac{\partial \Psi}{\partial t} \Psi \hat{H} \qquad \text{...(i)}$$

Where

$\hat{H}$ is Hamiltonion operator corresponding to the system and

Ψ is $\Psi(r,t)$

which is a complex valued wave function and describing the behaviour of system.

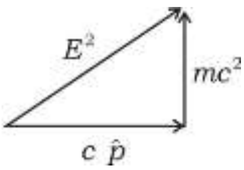
If, we include an additional discrete variable, spin quantum number(S), which is a non negative quantity the number 2S is an integer, odd for fermions and even for bosons. Each S has 2S + 1, Z-projection quantum numbers.

$$\sigma = S, S-1, \ldots\ldots 0, -S+1, \ldots\ldots\ldots -S$$

Therefore, wave function is now modify as $\Psi(r,t,\sigma)$ by the addition of σ, we can explain diverse range of subatomic particle behaviour and phenomenon: atomic configurations of atoms, nuclei andquark configuration and colour charge.

From special theory of relativity, the relativistic energy –momentum relation is expressed as;

$$E^2 = c^2 \hat{p} \cdot \hat{p} + (mc^2)^2 \qquad \text{...(ii)}$$



For relativistic wave equations are;

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \text{ and } \hat{p} = -i\hbar\nabla \qquad \text{...(iii)}$$

Therefore;

$$\left[\nabla^2 - \left(\frac{mc}{\hbar}\right)^2\right]\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \qquad \text{...(iv)}$$

The Heisenberg picture is an another formulation of quantum mechanics, in which the wave function is time independent and the operator $\hat{A}(t)$ contains the time dependence, governed by the equation of motion;

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}\left[\hat{A},\hat{H}\right] + \frac{\partial\hat{A}}{\partial t} \qquad \text{...(v)}$$

This equation is also valid in RQM, provided the Heisenbergoperators are modified to be consistent with special theory of relativity.

III. The Klein Gordan and Dirac equations for free particles:

The Klein Gordon equation is;

$$E^2\Psi = c^2\hat{p}\cdot\hat{p}\Psi + (mc^2)^2\Psi \qquad \text{...(vi)}$$

This is relativistically invariant, yet this equation alone is not a sufficient foundation for RQM due to a few reasons; one is thatnegative energy solution, another is density and this equation as it stand is only applicable to spinless particles.

Now; if factorised equation (i) as;

$$\left(\hat{E} - c\,\hat{\alpha}\cdot\hat{p} - \beta mc^2\right)\left(\hat{E} + c\,\hat{\alpha}\cdot\hat{p} + \beta mc^2\right)\Psi = 0 \qquad \text{...(vii)}$$

Where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are Hermiltian matrices those are required to anticommute for $i \neq j$

$$\alpha_i\beta = -\beta\alpha_i \quad \text{and} \quad \alpha_i\alpha_j = -\alpha_j\alpha_i$$

And square to the identify matrix

$$\alpha_i^2 = \beta^2 = I$$

Therefore; $(\hat{E} - c\hat{\alpha}\cdot\hat{p} - \beta mc^2)\Psi = 0$

$$\Rightarrow \hat{H} = c\,\hat{\alpha}\cdot\hat{p} + \beta\, mc^2 \qquad \text{...(viii)}$$

This is Dirac equation. The other equation

$$(E + c\hat{\alpha}\cdot\hat{p} + \beta mc^2)\Psi = 0$$

$$\Rightarrow \quad \hat{H} = (c\,\hat{\alpha}\cdot\hat{p} + \beta mc^2)$$

This equation

$\hat{H} = (c\,\hat{\alpha}\cdot\hat{p} + \beta mc^2)$ is also Dirac equation but for negative mass particle. Each factors are relativistically invariant.

IV. Spin and electromagnetic interacting particle

If $\dot{A}(\bar{r},t)$ is magnetic vector potential and $\phi(\underline{r},t)$ is electric scalar potential then

$$\vec{B} = \bar{\nabla}\times\vec{A}$$

$$\hat{E} \to \hat{E} - q\phi$$

$$\hat{p} \to \hat{p} - qA \rightleftharpoons \hat{p}_\mu \to \hat{p}_\mu - qA_\mu$$

While in non-relativistic QM

$$\hat{E} - q\phi = mc^2 \quad and \quad \hat{p} \cong mv$$

For spin zero (0) particle

In RQM, the K-G equation admits the minimal coupling preserptions;

The K-G equation is applicable to spinless charged bosons in an external electromagnetic potential.

For spin half (½) particle

Non relativistically, spin is introduced in the Pauli equationfor particles in an electromagnetic field.

$$\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi = \left[\frac{1}{2m}\left(\hat{\sigma}\left(\hat{p} - q\hat{A}\right)\right)^2\right] \quad ....(x)$$

By means of the 2 × 2 Pauli matrices, and Ψ is not just scalar wave function as in the non relativistic Schrödinger equation, but a two component spinor component

$$\Psi = \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}$$

where subscripts ↑ and ↓ refers to the spin up (+½) and spin down (-½) states.

In RQM, the Dirac equation can also incorporate minimal coupling;

$$\left(i\hbar\frac{\partial}{\partial t} - q\phi\right)\Psi = \gamma^{o}\left[C\gamma\cdot(\hat{p} - q\hat{A}) - mc^2\right]\Psi$$

$$= \left[\gamma^{\mu}\left(\hat{p}_\mu - q\hat{A}_\mu\right) - mc^2\right]\Psi$$

This is first equation to accurately predict spin, a consequence of the 4 × 4 gamma matrices.

$$\gamma^{o} = \beta, \gamma = (\gamma_1, \gamma_2, \gamma_3)$$
$$= \beta\alpha = (\beta\alpha_1, \beta\alpha_2, \beta\alpha_3)$$

Here ψ is a four component spinor field;

$$\psi = \begin{pmatrix} \Psi_{+} \\ \Psi_{-} \end{pmatrix} = \begin{pmatrix} \psi_{+}\uparrow \\ \psi_{+}\downarrow \\ \psi_{-}\uparrow \\ \psi_{-}\downarrow \end{pmatrix}$$

(V) Conclusion

Relativistic quantum mechanics has been developed and modify from early 20th century but after Poincare' covariant formulation its becomes very much useful. After inclusion of σ(z-projection of S) spin quantum state, the Dirac equation becomes very much useful.

Here in this approach we try our best to correlate RQM application for the explanation of spin association with fermions and bosons. Even helicity and chirality can be associate with above mentioned equation by simple definition and helicity for particle having mass, as;

$$\hat{h} = \hat{s} \cdot \frac{\hat{p}}{|\hat{p}|} = \hat{s} \cdot \frac{c\hat{p}}{\sqrt{E^2 - (mc^2)^2}}$$

and for massless particle, as

$$\hat{h} = \hat{s} \cdot \frac{c\ \hat{p}}{E}$$

Hence, in this article, we incorporate RQM with all states deciding parameters of particles.

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