A FIXED POINT RESULT OF R-WEAKLY COMMUTING MAPPING OF TYPE (A_f) VIA ALTERING DISTANCE FUNCTIONS

By

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Abstract:

Altering distance functions and their generalizations have been used to address an important class of fixed point problems. The main purpose of this paper is to obtain the conditions for the existence of a unique common fixed point for R- weakly commuting mapping of type (A_f) by using the altering distance function.

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1. Introduction:

A new category of contractive fixed point problems was addressed by Khan et.al. [8] and was extensively studied by several authors in [10],[11],[16],[17],[18] and [19].They introduced the concept of altering distances function which is actually a control function that alters distance between two points in a metric space and proved fixed point results for certain mappings.

Now a day it is very interesting tools to obtain the existence and uniqueness of common fixed point by altering distance function.

After the publication of Banach's contraction principle the next question that occupied the minds of mathematicians was to ascertain existence of simultaneous fixed points of more than one operator. It was observed in earlier literatures that apart from the inequality they may together satisfy, often commutativity is a requirement for the existence of common fixed points. Later efforts were made to relax this condition, which warranted a sequence of gradually weekend conditions like compatibility, commutativity, weakly commuting and R-weakly commuting conditions. In recent times R-weakly commuting conditions have been used in fixed point theory in a number of works like [12], [13], [14] and [21].

The view of the present paper is to obtain some common fixed point theorems by R- Weakly commuting mapping of type (A_f) by using the Altering distance function.

The study of common fixed point of non compatible mappings was initiated by Pant
[12] introduced the notion of point wise R- Weakly commutativity. Recently he established the point wise R-Weakly commutativity is a necessary condition for the existence of common fixed point theorem of contractive type pair of mappings.

The purpose of the present work is to combine the two lines of research. Specifically we prove here a fixed point results for two R-weakly commuting mappings that additionally satisfy an inequality, which involves altering distance function.

The following definitions are very helpful to study this paper

Definition1.1: Compatible Mapping [4]

Two selfmaps *A* and *S* of a metric space *(X, d)* are called **compatible** if $Lim_n \; d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n AX_n = \lim_n SX_n = t$ for some *t* in *X*.

Definition1.2: R-weakly commuting Mapping [14]

Two self maps *A* and *S* of a metric space *(X, d)* are called R-weakly commuting at a point *x* in *X* if $d(ASx, SAx) \le Rd(Ax, Sx)$ for *some_R* > 0

The maps *A* and *S* are called point wise **R-weakly commuting** on *X* if given *x* in *X* there exists $R > 0$ such that $d(ASx, SAx) \le Rd(Ax, Sx)$.

Definition1.3: R- weakly commuting of type (A_f) Mapping [14]

The mapping f and g are said to be **R- weakly commuting of type (A_f)** if there exists a positive real number R such that $d(fg(x), gg(x)) \le Rd(f(x), g(x))$ for all $x \in X$ *and* $R > 0$.

Then it is said to be R-weakly commuting of type (A_f).

Definition1.4: $\psi - R$ Weakly Commuting mapping:

Two self maps *A* and *S* of a metric space *(X, d)* are called $\psi - R$-weakly commuting at a point *x* in *X* if $\psi(\mathrm{d}\,(\mathrm{ASx}, \mathrm{SAx})) \le \mathrm{R}\psi(\mathrm{d}\,(\mathrm{Ax}, \mathrm{Sx}))$ for some R > 0.

Definition1.5: $\psi - R$ Weakly Commuting mapping of type (A_f):

The mapping f and g are said to be $\psi - R$ - weakly commuting of type (A_f) if there exists a positive real number R such that $\psi(d(fg(x), gg(x))) \le R\psi(d(f(x), g(x)))$ for all $x \in X$ *and* $R > 0$.

Then it is said to be $\psi - R$ -weakly commuting of type (A_f).

2. Main Theorem:

Theorem 2.1: Let f and g be R-weakly commuting self mapping of type (A_f) of a complete metric space (X,d) satisfying

1) $f(x) \subset g(x)$

2) $\psi(d(fx, fy)) \le \phi(\max(\psi(d(gx, gy), \psi(d(fx, gy), \psi(d(fy, gy))))$ ……….(1)

Where $\psi : R^+ \to R^+$ satisfying [1] the condition of altering distance function and $\phi : [0, \propto) \to [0, \propto)$ is a monotone increasing and continuous functions with the property that

i) $\phi(t) = 0$ if and only if t = 0

ii) $\phi(t) < t$.

for each t > 0. If f or g is continuous then f and g have a common fixed point.

Proof:

Let $y_n = f\,x_n = g\,x_{n+1}$ then from (1) $x = x_n$, $y = x_{n+1}$. we get

$$\psi(d(fx_n, fx_{n+1})) \le \phi(\max(\psi(d(gx_n, gx_{n+1}), \psi(d(fx_n, gx_{n+1}), \psi(d(fx_{n+1}, gx_{n+1}))))$$

$$\psi(d(y_n, y_{n+1})) \le \phi(\max(\psi(d(y_{n-1}, y_n)), \psi(d(y_n, y_{n-1})), \psi(d(y_{n+1}, y_n)))$$

$$\psi(d(y_n, y_{n+1})) \le \phi(\psi(d(y_{n-1}, y_n)))$$

$$d(y_n, y_{n+1})) < d(y_{n-1}, y_n)$$

Again putting

$x = x_{n-1}\ ,\ y = x_n.$ in (1)

$\psi(d(fx_{n-1}, fx_n)) \le \phi(\max(\psi(d(gx_{n-1}, gx_n), \psi(d(fx_{n-1}, gx_{n-1}), \psi(d(fx_n, gx_n))))$

$\psi(d(y_{n-1}, y_n)) \le \phi(\max(\psi(d(y_{n-2}, y_{n-1})), \psi(d(y_{n-1}, y_{n-2})), \psi(d(y_{n-1}, y_n)))$

$\psi(d(y_n, y_{n+1})) \le \phi(\psi(d(y_{n-2}, y_{n-1})))$

$d(y_n, y_{n-1})) < d(y_{n-2}, y_{n-1})$

We next show that $\{y_n\}$ is a Cauchy sequence .If not there exist an $\in$ >0 and two corresponding sub sequences $\{y_{n(k)}\} and \{y_{m(k)}\} of \{y_n\}$ such that for m (k)<n (k) $\therefore$ $d(y_{n(k)}, y_{m(k)}) \ge \in$ and $d(y_{n(k)-1}, y_{m(k)}) < \in$.

Again, $d(y_{n(k)}, y_{m(k)}) \le d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$

and $d(y_{n(k)-1}, y_{m(k)-1}) \le d(y_{n(k)-1}, y_{n(k)}) + d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1})$

Then we have $\in \le d(y_{n(k)}, y_{m(k)}) \le d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) < d(y_{n(k)}, y_{n(k)-1}) + \in$

making k $\to \propto$ we obtain

$\lim_{k\to\propto} d(y_{n(k)-1}, y_{m(k)-1}) = \in$ ……….(2)

also $d(y_{n(k)+1}, y_{m(k)-1}) \le d(y_{m(k)-1}, y_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(y_{n(k)}, y_{n(k)+1})$

$d(y_{n(k)}, y_{m(k)}) \le d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)-1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)})$

making $k \to \propto$ and using (2) we obtain

$\lim_{k\to\propto} d(y_{m(k)-1}, y_{n(k)+1}) = \in$ ………(3)

Lastly, $d(y_{n(k)}, y_{m(k)}) \le d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$

and $d(y_{n(k)}, y_{m(k)-1}) \le d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1})$

Making k $\to \propto$ and using (2) & (3) we obtain

$\lim_{k\to\propto} d(y_{n(k)}, y_{m(k)-1}) = \in .$ ………(4)

Now substituting $x = x_{n-1}\ ,\ y = x_{m-1}$ in (1) we have

$\psi(d(fx_{n-1}, fx_{m-1})) \le \phi(\max(\psi(d(gx_{n-1}, gx_{m-1}), \psi(d(fx_{n-1}, gx_{n-1}), \psi(d(fx_{m-1}, gx_{m-1}))))$

$\psi(d(y_{n-1}, y_{m-1})) \le \phi(\max(\psi(d(y_{n-2}, y_{m-2})), \psi(d(y_{n-1}, y_{n-2})), \psi(d(y_{m-1}, y_{m-2})))$

$\psi(\varepsilon) \le \phi(\max(\psi(\varepsilon), \psi(\varepsilon), \psi(\varepsilon)))$

$\psi(\varepsilon) \le \phi(\psi(\varepsilon)) < \psi(\varepsilon)$

which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete then there exist a point z in X such that $y_n \to z$ and also $y_n = fx_n = gx_n \to z$. Now

suppose that f is continuous. Then $ffx_n \to fz$ *and* $fgx_n \to fz$. Since f and g are $\psi - R$ -Weakly commuting of type (A_f).We have $\psi(d(fgx_n, ggx_n)) \le R\psi(d(fx_n, gx_n))$ .Letting $n \to \propto$ we obtain $ggx_n \to fz$. Now we prove that z = fz. If not then we have,

$\psi(d(fx_n, fgx_n)) \le \phi(\max(\psi(d(gx_n, ggx_n), \psi(d(fx_n, gx_n), \psi(d(fgx_n, ggx_n))))$

Taking $n \to \propto$ we obtain

$\psi(d(z, fz)) \le \phi(\max(\psi(d(z, fz), \psi(d(z, z), \psi(d(fz, fz))))$

$\psi(d(z, fz)) \le \phi(\max(\psi(d(z, fz)), \psi(0), \psi(0)))$ as $\psi(0) = 0$

$\psi(d(z, fz)) \le \phi(\psi(d(z, fz))) < \psi(d(z, fz))$. This is a contradiction.

Hence z = fz.

Since $f(X) \subset g(X)$ then there exist a point w in X such that z = fz = gw. Now if z = fw then from the inequality we get

$\psi(d(fgx_n, fw) \le \phi(\max(\psi(d(ggx_n, gw), \psi(d(fgx_n, ggx_n), \psi(d(fw, gw))))$

Taking $n \to \infty$ we obtain

$\psi(d(fz, fw)) \le \phi(\max(\psi(d(fz, gw)), \psi(d(fz, fz)), \psi(d(fw, gw))))$

$\psi(d(fz, fw)) \le \phi(\psi(d(fw, gw))) < \psi(d(fw, gw))$ which on letting $n \to \infty$ yields

$\psi(d(z, fw)) < \psi(d(fw, z))$

which is a contradiction. Therefore $z = fw$ and then $z = fz = gw = fw$. Hence z is a common fixed point of f and g. The proof is similar when g is assumed to be continuous.

Examples:

Let X=[2,10] be a metric space with usual metric d. Consider $\psi : R^+ \to R^+$ be defined as $\psi(x) = x^2$ Also define

$$g(x) = \begin{cases} 2; \; x = 2 \\ x^2 - 5x + 8; \; x > 2 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 2; \; x = 2 \\ \frac{x}{2} + 1; \; x > 2 \end{cases}.$$

Here we define $\{X_n\}$by $X_n=2+\frac{1}{n}$ and we see that $f(x_n) = g(x_n) = 2$ *also* $fg(x_n) = gf(x_n) = 2$

Also it satisfy the inequality (1).

Calculation: when x = 2,y=2 $\psi(d(fx, fy)) = (fx - fy)^2 = 0$.

when x>2 and y>2; $\psi(d(fx, fy)) = (fx - fy)^2 = (x - y)^2 / 4$

$$\psi(d(fx, gy)) = (2y^2 - 10y + 14 - x)^2 / 4$$

$$\psi(d(fy, gy)) = (fy - gy)^2 = (2y^2 - 11y + 14)^2 / 4$$

$$\psi(d(gx, gy)) = (gx - gy)^2 = [(x - y)(x + y - 5)]^2$$

When x=y then it is obvious. Hence it satisfies all the conditions of our theorem 2.1 .

R.H.S: We now calculate,

$$\max\left\{\frac{(2y^2 - 10y - x + 14)^2}{4}, \frac{(2y^2 - 11y + 14)^2}{4}, ((x - y)(x + y + 5))^2\right\} \ldots\ldots\ldots\ldots(A)$$

Now if we take $\frac{(2y^2 - 10y - x + 14)^2}{4} > \frac{(2y^2 - 11y + 14)^2}{4}$

Then $2y^2 - 10y - x + 14 > 2y^2 - 11y + 14$ for all x, y$\in[2,10]$, implies *y>x.*

Let, ($2y^2 - 11y + 14$)/2> $x^2 - y^2 - 5x + 5y$ or, $4y^2 - 2x^2 - 21y + 10x + 14 > 0$

or, $4y^2 - 2x^2 - 20y + 9x + 14$+(x-y) >0 …………….(a)

Again suppose $(2y^2 - 10y - x + 14)/2 < x^2 - y^2 - 5x + 5y$

or, $(2y^2 - 10y - x + 14) < 2x^2 - 2y^2 - 10x + 10y$

implies $4y^2 - 2x^2 - 20y + 9x + 14 < 0$ ...……………..(b)

From relation (a) & (b) we get (x - y)>0 implies $x > y$

Hence for $x > y$ maximum value of (A) becomes $\frac{(2y^2 - 11y + 14)^2}{4}$.

L.H.S : For $(x - y)^2/4$, we have $(x - y)^2 / 4 < \frac{(2y^2 - 11y + 14)^2}{4}$ where $x, y \in X \in [2,10]$.

Therefore it satisfies all the conditions of **Theorem 2.1.** Thus in view of theorem 2.1, *f* and *g* have a common fixed point say *x=2.*

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