Some Properties of Minimal and Maximal Open Sets in Alexandroff Space

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Abstract: In this paper we have studied the minimal open sets and maximal open sets in the Alexandroff space or σ-space. Also studied the relation between minimal open sets and preopen sets in the Alexandroff space or σ-space.

Keyword: Alexandroff space or σ-space, minimal open sets, maximal open sets, preopen sets.

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1. INTRODUCTION:

The Notion of a topological space was generalized to a σ-space(or Alexandroff space or simply space) by A. D. Alexandroff[1] weakening the union requirements. A field of study remains vivid and investigative till new contributions to the theory of the field are remain to add constantly. General topology is an example of such a field and its theory are enriching day by day by adding new contributions from various directions. According to such trends Nakaoka and Oda [7] introduced and studied the concept of minimal open sets in a topological space. By dualizing the concept of minimal open sets, Nakaoka and Oda [8] introduced and studied the idea of maximal open sets. Some authors also made investigation about the minimal open set and maximal closed sets in different directions.The notion of pre-open set in topological space was introduced by Mashhour et al. [5]. Banerjee and Saha[2] introduced the concept of preopen sets in σ-space or Alexandroff space in two different ways. In this paper we wish to study the idea of minimal open sets and maximal open sets in more general structureof a Alexandroffspace or σ-space. We have also investigated here how far several results as valid intopological space are acted in a σ-space. In this paper we study fundamental properties of minimal open sets and apply them to obtain some results on pre-open sets.

2. PRELIMINARIES:

Definition 2.1[1]: A set X is called an Alexandroff space or simply a space if in it is chosen a system F of subsets satisfying the following axioms:

1) The intersection of a countable number of sets from F is a set in F.
2) The union of a finite number of sets from F is a set in F.
3) The void set φ is a set in F.
4) The whole set X is a set in F.

Sets of F are called closed sets. Their complementary sets are called open. It is clear that instead of closed sets in the definition of the space one may put open sets with subject to the conditions of countable summability, finite intersectibility and the condition that X and void set φ should be open. The collection of all such open sets will sometimes be denoted by τ and the space by (X,τ). Note that a topological space is a space but in general τ is not a topology as can be easily seen by taking $X = R$ and τ as the collection of all F_σ sets in R.

Definition 2.2[1]: To every set M of a space (X,τ) we correlate its closure $\bar{M}$, the intersection of all closed sets containing M. The closure of a set M will be denoted by $\tau cl(M)$ or simply clM when there is no confusion about τ.

Generally, the closure of a set in a space may not be a closed set. The definition of limit point of a set is parallel as in the case of a topological space.

From the axioms, it easily follows that

1) $\overline{M \cup N} = \bar{M} \cup \bar{N}$;
2) $M \subset \bar{M}$;
3) $\bar{M} = \bar{\bar{M}}$;
4) $\bar{\varphi} = \varphi$.
5) $\bar{A} = A \cup A'$

where A' denotes the set of all limit point of A.

Definition 2.3[4]: The interior of a set M in a space (X,τ) is defined as the union of all open sets contained in M and is denoted by τ-int(M) or int(M) when there is no confusion about τ.

Definition 2.4 [5]: Let (X,τ) be a topological space. A subset A of X is said to be preopen if

$A \subset int(cl(A))$.

3. Minimal Open sets:

Let (X,τ) be a σ-space.

Definition 3.1: A nonempty open set U of σ-space $X$is said to be minimal open set if and only if any open set which is contained in U is $\emptyset$ or U.

Lemma 3.2: (1) Let U be a minimal open set and Wan open set. Then$U \cap W = \emptyset$or$U \subset W$.

(2) Let U and V are two minimal open sets. Then either$U \cap V = \emptyset$ or$U = V$.

Proof:(1) Let W be an open set such that $U \cap W \neq \emptyset$. Then we have to see that since U is a minimal open set and $U \cap W \subset U$, so the only opportunity is that $U \cap W = U$. Therefore, $U \subset W$.

(2) Consider that $U \cap V \neq \emptyset$, then by above (1) and as U and V are two minimal open sets we get, $U \subset V$ and $V \subset U$. Therefore, $U = V$.

Proposition 3.3: Let x is an element of a minimal open set U of σ-space X. Then for any open neighborhood W of x, $U \subset W$.

Proof: Let W be an open neighborhood of x such that $U \not\subset W$. Then $U \cap W$ is an open set such that $U \cap W \subsetneq U$ and $U \cap W \neq \emptyset$. Since U is a minimal open set we arise a contradiction. Therefore, the only possibility is that $U \subset W$.

Proposition 3.4: Let U be a minimal open set of σ-space X. Then$U = \cap \{W | W$ is an open neighborhood of $x\}$

for any element of x of U.

Proof: By the above proposition and as U is an open neighborhood of x, we can write$U \subset \cap \{W | W$ is an open neighborhood of $x\} \subset U$.

Therefore, the result is obtained.

Theorem 3.5: Let U be a nonempty open set of σ-space X. Then the following three conditions are equivalent:

(1) U is a minimal open set.

(2) $U \subset Cl(S)$ for any nonempty subset S of U.

(3) $Cl(U) = Cl(S)$ for any nonempty subset S of U.

Proof: (1) ⇒ (2) Let S be any nonempty subset of U. By proposition 3.3, for any $x \in U$ and any open neighborhood W of x, we have $S = U \cap S \subset W \cap S$. This implies that $W \cap S \neq \emptyset$ and hence x is an element of $Cl(S)$. It follows that $U \subset Cl(S)$.

(2) ⇒ (3). Let S be any nonempty subset of U, then obviously $Cl(S) \subset Cl(U)$. Again, since $U \subset Cl(S)$ (by (2)) we get $Cl(U) \subset Cl(Cl(S)) = Cl(S)$. Therefore, we conclude that $Cl(U) = Cl(S)$ for any nonempty subset S of U.

(3) ⇒ (1). Now we prove this by contradiction. Assuming that U is not a minimal open set. Then by the property of minimal open set there exists a nonempty open set V such that $V \subsetneq U$. Then there must exists an element $a \in U$ such that $a \notin V$. Thus the closure of $\{a\}$ contains in the complement of V i.e., $Cl(\{a\}) \subset V^c$, where V^c is the complement of V. It follows that $Cl(\{a\}) \neq Cl(U)$, which contradicts (3). Hence the result.

Theorem 3.6: Let U be a minimal open set and x an element of $X - U$. Then $W \cap U = \emptyset$ or$U \subset W$ for any open neighborhood W of x.

Proof: Since W is an open set, we have the result by Lemma 3.2.

Corollary3.7: Let U be a minimal open set and x an element of $X - U$. Define $U_x = \cap \{W : W \text{ is an open neighborhood of } x\}$. Then $U_x \cap U = \emptyset$ or $U \subset U_x$.

Proof: If $U \subset W$ for any open neighborhood W of x, then $U \subset \cap \{W : W \text{ is an open neighborhood of } x\}$. Therefore$U \subset U_x$. In the other hand there exists an open neighborhood W of x such that $W \cap U = \emptyset$. Then we have $U_x \cap U = \emptyset$.

4. Preopen sets

In a topological space the following conditions (I) and (II) are equivalent.

Condition (I): $A \subset int(cl(A))$ and

Condition (II): there exists an open set U such that $A \subset U \subset cl(A)$.

But in a σ-space (Alexandroff space) these two condition are not equivalent. In fact, condition (I) is weaker than the condition (II) as shown in the example 4.2. In view of above observation, we take condition (II) to define preopen sets in a space.

Definition 4.1[2]: Let (X, τ) be a space. A subset A of X is said to be preopen if there exists an open set U such that $A \subset U \subset cl(A)$ and A is said to be weakly preopen if

$A \subset int(cl(A))$.

Note that if A is open then $int(A) = A$. So $A = int(A) \subset int\big(cl(A)\big)$ and hence A is weakly preopen. Also, if A is open then the condition (II) holds if we take $U = A$. So, every open set is preopen also. But converse may not be true as shown in the following example:

Example 4.2: Let $X = [1,2]$ and $\tau = \{X, \varphi, F_i\}$ where F_i's are the countable subsets of irrational in [1,2]. Let $A = [1,2] - Q$. Then $cl(A) = X$. So A is weakly preopen, since condition (I) holds. Also, condition (II) holds if we take $U = X$. Hence A is preopen also. But A is not open. Next, let $B = ([1,2] - Q) - \{\sqrt{2}\}$. Then $cl(B) = X - \{\sqrt{2}\}$ and $int\big(cl(B)\big) = B$. Therefore, $B \subset int(cl(B))$. Again, there does not exists any τ-open set U such that $B \subset U \subset cl(B)$. So B is weakly preopen but not preopen.

Theorem 4.3: Let U be a minimal open set. Then any nonempty subset S of U is apreopen set.

Proof: By using theorem3.5(2), we have $Int(U) \subset Int\big(Cl(S)\big)$. Since U is an open set, we have $S \subset U = Int(U) \subset Int\big(Cl(S)\big)$.

Theorem 4.4: Let U be a minimal open set and M a nonempty subset of σ-space X. If there exists an open neighborhood $W$of M such that $W \subset Cl(M \cup U)$, then $M \cup S$ is a preopen set for any nonempty subset S of U.

Proof : By theorem 3.5(3), we have $Cl(M \cup S) = Cl(M) \cup Cl(S) = Cl(M) \cup Cl(U) = Cl(M \cup U)$. Now by assumption, $W \subset Cl(M \cup U) = Cl(M \cup S)$. Again $U \subset Cl(U) \subset Cl(U) \cup Cl(M) = Cl(M \cup U) = Cl(M \cup S)$. Since W is open neighborhood of M, it is a open set. So $W \cup U$ is open set. Now $M \cup S \subset W \cup U \subset Cl(M \cup S)$. This implies that $M \cup S$ is preopen set.

Corollary 4.5: Let U be a minimal open set and M a nonempty subset ofσ-space X. If there exists an open neighborhood W of M such that $W \subset Cl(U)$, then $M \cup S$ is a preopen set for any nonempty subset S of U.

Theorem 4.6: Let U be a minimal open set and x an element of $X - U$. Then $W \cap U = \emptyset \; or \; U \subset W$ for any neighborhood W of x.

4.1 Finite open sets.

Now we study some properties of minimal open sets in finite open sets and locally finite spaces in the σ- space or Alexandroff space.

Theorem 4.1.1. Let V be a nonempty finite open set. Then then exists at least one(finite) minimal open set U such that $U \subset V$.

Proof: If V is a minimal open set, we may set $U = V$. If V is not a minimal open set, then there exists an (finite) open set $V_1(\neq \emptyset)$ such that $V_1 \subsetneq V$. If V_1 is not a minimal open set, then there exists an (finite) open set $V_2(\neq \emptyset)$ such that $V_2 \subsetneq V_1 \subsetneq V$. Continuing this process, we have a sequence of open sets $V \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_k \supsetneq \cdots$. Since V is finite set, this process continues up to finite steps. Therefore, after finite steps we finally get a minimal open set V_n such that $U = V_n$, for some positive integer n.

Definition 4.1.2: A σ- space or Alexandroff space is said tobe a locally finite space if each of its elements is contained in a finite open set.

Corollary 4.1.3: Let X be a locally finite σ- space and V a nonempty open set. Then there exists at least one (finite) minimal open set U such that $U \subset V$.

Proof: Since V is a nonempty set, there exists an element x of V. Again since X is a locally finite σ- space, we can find a finite open set V_x which containing x i.e, $x \in V_x$. Since $V \cap V_x$ is a finite open set, we get a minimal open set U such that

$$U \subset V \cap V_x \subset V.$$

Theorem 4.1.4: Let V_i be an open set for $i \in \Lambda$ and W a nonempty finite open set. Then $W \cap (\cap_{i\in\Lambda} V_i)$ is a finite open set.

Proof: We see that there exists an integer n such that

$$W \cap (\cap_{i\in\Lambda} V_i) = W \cap (\cap_{k=1}^{n} V_{i_k})$$

and hence we get the result.

Now the

Theorem 4.1.5: Let V_i be an open set for $i \in \Lambda$ and W_α a nonempty finite open set for any $\alpha \in \Omega$, where Ω is countable index set. Let $S = \cup_{\alpha\in\Omega} W_\alpha$. Then $S \cap (\cap_{i\in\Lambda} V_i)$ is an open set.

Proof: Since W_α is a finite open set, we have

$$W \cap (\cap_{i\in\Lambda} V_i)$$

is a finite open set for any $\alpha \in \Omega$. Since

$S \cap (\cap_{i\in\Lambda} V_i) = (\cup_{\alpha\in\Omega} W_\alpha) \cap (\cap_{i\in\Lambda} V_i) = \cup_{\alpha\in\Omega} (W_\alpha \cap (\cap_{i\in\Lambda} V_i))$,

We have the result.

5. MAXIMAL OPEN SETS:

Now we have to studied the maximal open set in Alexandroff space. A proper nonempty subset U of X is said to be maximal open set if any open set containing U is either X or U it self. Therefore, there are no other open set except X or U containing U. Though this definition of maximal open set is obtained by dualizing the definition of minimal open set, but there are some properties which is not obtained by dualizing the properties of minimal open set. This is the importance of the study of the maximal open set.

Definition 5.1: A proper open subset U of σ- space or Alexandroff space(X, τ) is said to be a maximal open set if any open set containing U will be itself or X.

Theorem 5.2: (1) Let U be a maximal open set and W an open set. Then

$U \cup W = X$ or $W \subset U$.

(2) Let U and V be maximal open sets. Then

$U \cup V = X$ or $U = V$

Proof: (1) Let W be an open set such that $U \cup W \neq X$. Since U is maximal open set and $U \subset U \cup W$, we get $U \cup W = U$. Therefore, $W \subset U$.

(2) If $U \cup V \neq X$, then $V \subset U$ and $U \subset V$. Therefore, $U = V$.

Theorem 5.3: Let U_α, U_β and U_γ be a maximal open sets in Alexandroff space such that $U_\alpha \neq U_\beta$. If $U_\alpha \cap U_\beta \subset U_\gamma$, then $U_\alpha = U_\gamma$ or $U_\beta = U_\gamma$.

Proof: We see that

$U_\alpha \cap U_\gamma = U_\alpha \cap (U_\gamma \cap X)$

$= U_\alpha \cap (U_\gamma \cap (U_\alpha \cup U_\beta))$, since U_α, U_β are disjoint maximal open sets.

$= U_\alpha \cap ((U_\gamma \cap U_\alpha) \cup (U_\gamma \cap U_\beta))$

$= (U_\alpha \cap U_\gamma) \cup (U_\alpha \cap U_\gamma \cap U_\beta)$

$= (U_\alpha \cap U_\gamma) \cup (U_\alpha \cap U_\beta)$, Since $U_\alpha \cap U_\beta \subset U_\gamma$.

$= U_\alpha \cap (U_\gamma \cup U_\beta)$.

Hence we have $U_\alpha \cap U_\gamma = U_\alpha \cap (U_\gamma \cup U_\beta)$. If $U_\gamma \neq U_\beta$, then $U_\gamma \cup U_\beta = X$, and hence $U_\alpha \cap U_\gamma = U_\alpha$; namely, $U_\alpha \subset U_\gamma$. Since U_α and $U_\gamma$are maximal open sets, we have $U_\alpha = U_\gamma$.

Theorem 5.4: Let U_α, U_β and U_γ be a maximal open sets in Alexandroff space where they are different from each other. Then,

$U_\alpha \cap U_\beta \not\subset U_\alpha \cap U_\gamma$.

Proof: If $U_\alpha \cap U_\beta \subset U_\alpha \cap U_\gamma$, then we see that

$$(U_\alpha \cap U_\beta) \cup (U_\beta \cap U_\gamma) \subset (U_\alpha \cap U_\gamma) \cup (U_\beta \cap U_\gamma)$$

Hence,

$$U_\beta \cap (U_\alpha \cup U_\gamma) \subset (U_\alpha \cup U_\beta) \cap U_\gamma$$

Since $U_\alpha \cup U_\gamma = X = U_\alpha \cup U_\beta$, we have $U_\beta \subset U_\gamma$. It follows that $U_\beta = U_\gamma$, which contradicts our assumption.

Theorem 5.5: Let V be a proper nonempty cofinite open subset of σ-space or Alexandroff space. Then, there exists, at least , one (cofinite) maximal open set U such that $V \subset U$.

Proof: If V is a maximal open set, we may set $V = U$. If V is not a maximal open set, then there exists an (cofinite) open set V_1 such that $V \subsetneq V_1 \neq X$. If V_1 is a maximal open set, we may set $V_1 = U$. If V_1 is not a maximal open set, then there exists an (cofinite) open set V_2 such that $V \subsetneq V_1 \subsetneq V_2 \neq X$. Continuing this process, we have a sequence of open sets

$$V \subsetneq V_1 \subsetneq V_2 \dots \subsetneq V_k \subsetneq \cdots$$

Since V is a cofinite set, this process repeats only finitely. Then, finally, we get a maximal open set V_n such that $V_n = U$, for some positive integer n.

6. Closure, interior and maximal open sets.

Theorem 6.1: Let U be a maximal open set in Alexandroff space X and x an element of $X - U$. Then $X - U \subset W$ for any open neighborhood W of x.

Proof: Since $x \in X - U$ and $W \not\subset U$ for any open neighborhood W of x. Then, $W \cup U = X$. Therefore, $X - U \subset W$.

Corollary 6.2: Let U be a maximal open set in Alexandroff space X. Then either of the following (1) and (2) holds:

(1) For each $x \in X - U$ and each open neighborhood W of x, $W = X$.

(2) There exists an open set W such that $X - U \subset W$ and $W \subsetneq X$.

Proof: If (1) does not hold, then there exists an element x of $X - U$ and an open neighborhood W of x such that $W \subsetneq X$. Then by above theorem we have $X - U \subset W$.

Corollary 6.3: Let U be a maximal open set in Alexandroff space X. Then either of the following (1) and (2) holds:

(1) For each $x \in X - U$ and each open neighborhood$W$ of x, we have $X - U \subsetneq W$;

(2) There exists an open set W such that $X - U = W \neq X$.

Proof: Assume that (2) does not hold. Then by above theorem we have $X - U \subset W$ for each $x \in X - U$ and each open neighborhood W of x. Hence, we have $X - U \subsetneq W$.

Theorem 6.4: Let U be a maximal open set in Alexandroff space X. Then $Cl(U) = X$ or $Cl(U) = U$.

Proof: Since U be a maximal open set in Alexandroff space X, by above corollary either of the following cases (1) and (2) occur:

(1) For each $x \in X - U$ and each open neighborhood$W$ of x, we have $X - U \subsetneq W$: let x be any element of $X - U$ and W be any open neighborhood of x. Since $X - U \neq W$, we have $W \cap U \neq \emptyset$ for any open neighborhood W of x. Hence, $X - U \subset Cl(U)$. Since $X = U \cup (X - U) \subset U \cup Cl(U) = Cl(U) \subset X$, we have $Cl(U) = X$;

(2) There exists an open set W such that $X - U = W \neq X$: Since $X - U = W$ is an open set, U is a closed set. Therefore, $U = Cl(U)$.

Theorem 6.5: Let U be a maximal open set in Alexandroff space X. Then, $Int(X - U) = X - U$ or $Int(X - U) = \emptyset$.

Proof: By corollary 6.3, we have either (1) $Int(X - U) = \emptyset$ or by (2) $Int(X - U) = X - U$.

Theorem 6.6: Let U be a maximal open set in Alexandroff space X and S a nonempty subset of $X - U$. Then, $Cl(S) = X - U$.

Proof: Since $S \subset X - U$, we have $W \cap S \neq \emptyset$ for any open neighborhood W of $x \in X - U$. Then $X - U \subset Cl(S)$. Since $X - U$ is a closed and $S \subset X - U$, we see that $Cl(S) \subset Cl(X - U) = X - U$. Therefore $Cl(S) = X - U$.

Corollary 6.7: Let U be a maximal open set in Alexandroff space X and M a nonempty subset of X where $U \subsetneq M$. Then, $Cl(M) = X$.

Proof: Since $U \subsetneq M \subset X$ there exists a nonempty subset S of $X - U$ such that $M = U \cup S$. Hence we have $Cl(M) = Cl(U \cup S) = Cl(U) \cup Cl(S) \supset U \cup (X - U) = X$, by theorem 6.6. Therefore, $Cl(M) = X$.

Theorem 6.8: Let U be a maximal open set in Alexandroff space X and assume that the subset $X - U$ has at least two elements. Then, for any element a of $X - U$, $Cl(X - \{a\}) = X$.

Proof: Since $U \subsetneq X - \{a\}$ by our assumption, we have the result by corollary 6.7.

Theorem 6.9: Let U be a maximal open set in Alexandroff space X and N a proper subset of X with $U \subset N$. Then, $Int(N) = U$.

Proof: If $N = U$, then $Int(N) = Int(U) = U$. Otherwise $N \neq U$, and hence $U \subsetneq N$. It follows that $U \subset Int(N)$. Since U is maximal open set, we have also $Int(N) \subset U$. Therefore, $Int(N) = U$.

Theorem 6.10: Let U be a maximal open set in Alexandroff space X and S a nonempty subset of $X - U$. Then, $X - Cl(S) = Int(X - S) = U$.

Proof: Since $U \subset X - S \subsetneq X$ by our assumption, we get the result by the theorems 6.6 and 6.9.

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