PERMANENT OF INTERVAL-VALUED AND TRAPEZOIDAL FUZZY NUMBER MATRICES

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ABSTRACT: ***In this paper, permanent of both interval-valued fuzzy matrices (IVFMs) and trapezoidal number fuzzy matrices are defined with examples. Some properties due to permanent nature of IVFMs are presented here. Properties related to the permanent of IVFMs and TrFMS are proved. Also a method to evaluate the permanent for large order IVFM is described.***

Keywords : Fuzzy matrices, Interval-valued fuzzy matrices (IVFMs), Fuzzy Number, Trapezoidal Fuzzy Number (TrFNs), Trapezoidal fuzzy number matrices (TrFMs), Permanent.

1 INTRODUCTION

The Permanent has a rich structure when restricted to certain classes of matrices, particularly, matrices of zeors and ones, (entrywise) nonnegative matrices and positive semidefinite matrices. Furthermore, there is a certain similarly of its properties over the class of nonnegative matrices and the class of positive semidefinite matrices.

Fuzzy sets have been introduced by Lofti.A.Zadeh[13] Fuzzy set theory permits the gradual assessments of the membership of elements in a set which is described in the interval [0,1]. It can be used in a wide range of domains where information is incomplete and imprecise. Interval arithmetic was first suggested by Dwyer [2] in 1951, by means of Zadeh's extension principle. A fuzzy number is a quantity whose values are imprecise, rather than exact as is the case with single – valued numbers

The concept of Rank of a matrix with fuzzy numbers as its elements, which may be used to modern uncertain imprecise aspects of real-word problems. We studied main ideas based on rank of fuzzy matrix and arithmetic operations. We give some necessary and sufficient conditions for algorithm to find rank of fuzzy matrices based on Triangular fuzzy number. In Dubosis and Prade [1] arithmetic operations will be employed for the same purpose but with respect to the inherent difficulties which are derived from the positively restriction on Triangular fuzzy number. C.jaisankar and S.Arunvasan [5] proposed concept of Rank and Nullity of a triangular fuzzy matrix and some of the relevant theorems will be revalued. Finally fuzzifing the defuzzified version of the original problem for introducing fuzzy rank. E.Romanowicz and A.Grabowski [7] proposed on the permanent of a matrix. M.pal [8] proposed intuitionistic fuzzy linear transformation.

The fuzzy matrices introduced first time by Thomson [11] and discussed about the convergence of powers of fuzzy matrix. C.jaisankar and S.Arunvasan [5] presented some important results on rank and nullity of triangular fuzzy number matrices. A.K.Shyamal and M.Pal [9,10] first time introduced triangular fuzzy matrices . In C.Jaisankar and S.Arunvasan [3,4] introduced the concept on Hessenberg of Triangular Fuzzy Matrices. C.Jaisankar and S.Arunvasan [6] the notion called adjoint of matrix have been defined with the domain of triangular fuzzy number. We state a formula for the adjoint matrix of a square triangular fuzzy matrix and this formula shall be used .

The paper organized as follows, Firstly in section 2, we recall the basic definitions of Triangular and Trapezoidal fuzzy number and some operations on triangular and trapezoidal fuzzy numbers (TFNs) and (TrFNs). In section 3, we defined permanent of IVFMs. In section 4, we defined evaluation of permanent of large order IVFM. In section 5, we have been presented the permanent of TrFMs with the aid of notion. Finally in section 6, conclusion is included.

I. PRELIMINARIES

In this section, some basic definition of FMs, IVFMs, TrFNs, TrFMs, and permanent of crisp matrix are given. IVFM denotes the set of all interval-valued fuzzy matrices, that is, fuzzy matrices whose entries are all subintervals of the interval [0,1].

Definition 2.1 fuzzy set

A fuzzy set is characterized by a membership function mapping the element of a domain, space or universe of discourse X to the unit interval [0,1]. A fuzzy set A in a universe of discourse X is defined as the following set of pairs

$$A=\{(x,\mu_A(x)) ; x\in X\}$$

Here $\mu_A : X \to [0,1]$ is a mapping called the degree of membership function of the fuzzy set A and $\mu_A(x)$ is called the membership value of $x\in X$ in the fuzzy set A. These membership grades are often represented by real numbers ranging from [0,1].

Definition 2.2 Normal fuzzy set

A fuzzy set A of the universe of discourse X is called a normal fuzzy set implying that there exists at least one $x\in X$ such that $\mu_A(x) =1$.

Definition 2.3 Convex fuzzy set

A fuzzy set $A=\{(x,\mu_A(x))\}\subseteq X$ is called Convex fuzzy set if all $A_\propto$ are Convex set (i.e.,) for every element $x_1\in A_\propto$ and $x_2\in A_\propto$ for every $\alpha\in[0,1]$, $\lambda x_1+(1-\lambda)\, x_2\in A_\alpha$ for all $\lambda\in[0,1]$ otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.4 Fuzzy Matrix

A fuzzy matrix A of order $m \times n$ is defined by $A = [< a_{ij}, a_{ij\mu} >]_{m\times n}$ where $a_{ij\mu}$ is the membership value of the element a_{ij} $in\ A$ and $a_{ij\mu} \in [0,1]$.

For simplicity, we write A as $A = [a_{ij\mu}]_{m\times n}$.

For a pair of fuzzy matrices $E = [e_{ij}]$ and $F = [f_{ij}]$ $in\ F_{mn}$ such that $E \leq F$,let us define the interval matrix as $[E, F]$, whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}, that is $[e_{ij}, f_{ij}]$. In particular, for $E = F, IVFM[E, F]$ reduce to the fuzzy matrix $E \in F_{mn}$.

Definition 2.5 Interval-Valued Fuzzy Matrix

An IVFM A over F_{mn} is defined as $A = [a_{ij}] = [[a_{ijL}, a_{ijU}]]_{m\times n}$. Let us define $A_L = [a_{ijL}]_{m\times n}$ $and\ A_U = [a_{ijU}]_{m\times n}$, clearly A_L $and\ A_U$ belong to F_{mn} such that $A_L \leq A_U$. In short, A can be written as $A = [A_L, A_U]$, where A_L $and\ A_U$ are called the lower and upper limits of A, respectively. For simplicity, we write an IVFM as $A = [[a_{ijL}, a_{ijU}]]_{m\times n}$ with maintaining the condition $0 \leq a_{ijL} \leq a_{ijU} \leq 1$.

Here we shall follow the basic operations on IVFM.

Let $A = [a_{ij}] = [[a_{ijL}, a_{ijU}]]$ $and\ B = [b_{ij}] = [[b_{ijL}, b_{ijU}]]$ be two IVFMs of order $m \times n$, their sum is denoted by $A + B$ and is defined as

$$A + B = [\max\{a_{ij}, b_{ij}\}] = [\max\{a_{ijL}, b_{ijL}\}, \max\{a_{ijU}, b_{ijU}\}].$$

Product of two IVFMs $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ denoted by AB and is defined as

$$AB = [c_{ij}] = [\textstyle\sum_{k=1}^{n} a_{ik}b_{kj}], i = 1,2, \dots, m\ and\ j = 1,2, \dots, p$$
$$= [\textstyle\sum_{k=1}^{n}(a_{ikL}.b_{kjL}),\ \ \sum_{k=1}^{n}(a_{ikU}.b_{kjU})]$$
$$= [\max_k \min(a_{ikL}.b_{kjL}), [\max_k min\ (a_{ikU}.b_{kjU})].$$

If $A = [A_L, A_U]$ $and\ B = [B_L, B_U]$, then $A + B = [A_L + B_L,\ A_U + B_U]$, $AB = [A_LB_L,\ A_UB_U]$.

$A \geq B$ if and only if $a_{ijL} \geq b_{ijL}$ and $a_{ijU} \geq b_{ijU}$. Also, it can be proved that $A \geq B$ if and only if $A + B = A$.

Definition 2.6 Fuzzy number

A fuzzy set $\tilde{A}$ defined on the set of real number R is said to be fuzzy number if its membership function has the following characteristics

i. $\tilde{A}$ is normal
ii. $\tilde{A}$ is convex
iii. The support of $\tilde{A}$ is closed and bounded then $\tilde{A}$ is called fuzzy number.

Definition 2.7 Trapezoidal fuzzy number

A fuzzy number $\tilde{A}^{TzL} = (a_1, a_2, a_3, a_4)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}^{TzL}}(x) = \begin{cases} 0 & ;\ x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & ;\ a_1 < x \leq a_2 \\ 1 & ;\ a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & ;\ a_3 < x \leq a_4 \\ 0 & ;\ x \geq a_4 \end{cases}$$

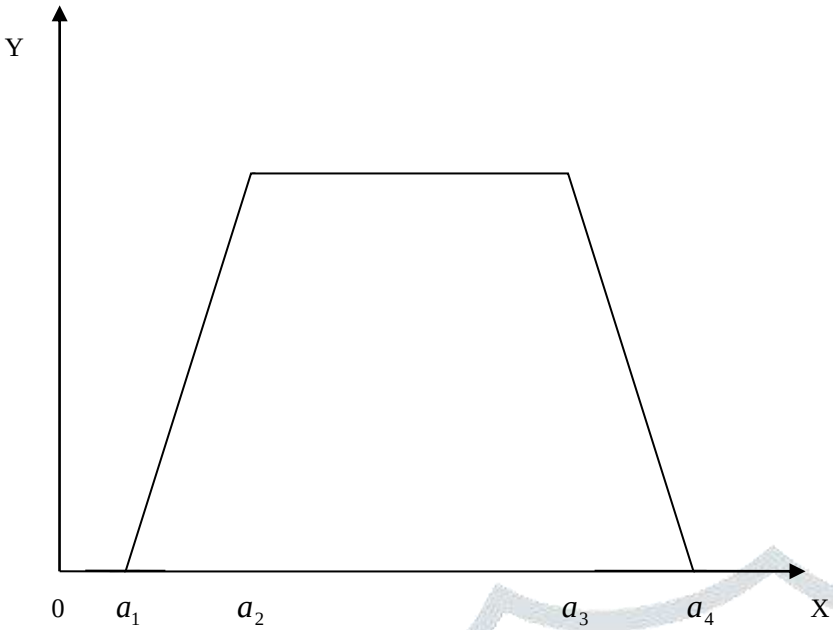

Fig:1 Trapezoidal Fuzzy Number

Definition 2.8 Ranking function

We defined a ranking function $\Re$: F(R)→R which maps each fuzzy numbers to real line F(R) represent the set of all trapezoidal fuzzy number. If R be any linear ranking function

$$\Re(\tilde{A}^{TzL}) = \left(\frac{a_1 + a_2 + a_3 + a_4}{4}\right)$$

Also we defined orders on F(R) by

$\Re(\tilde{A}^{TzL}) \geq \Re(\tilde{B}^{TzL})$ if and only if $\tilde{A}^{TzL} \geq_R \tilde{B}^{TzL}$

$\Re(\tilde{A}^{TzL}) \leq \Re(\tilde{B}^{TzL})$ if and only if $\tilde{A}^{TzL} \leq_R \tilde{B}^{TzL}$

$\Re(\tilde{A}^{TzL}) = \Re(\tilde{B}^{TzL})$ if and only if $\tilde{A}^{TzL} =_R \tilde{B}^{TzL}$

Definition 2.9 Arithmetic operations on trapezoidal fuzzy numbers (TrFNs)

Let $\tilde{A}^{TzL} = (a_1, a_2, a_3, a_4)$ and $\tilde{B}^{TzL} = (b_1, b_2, b_3, b_4)$ be trapezoidal fuzzy numbers (TrFNs) then we defined,

Addition

$$\tilde{A}^{TzL} + \tilde{B}^{TzL} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

Subtraction

$$\tilde{A}^{TzL} - \tilde{B}^{TzL} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

Multiplication

$$\tilde{A}^{TzL} \times \tilde{B}^{TzL} = (a_1\Re(B), a_2\Re(B), a_3\Re(B), a_4\Re(B))$$

where $\Re(\tilde{B}^{TzL}) = \left(\frac{b_1 + b_2 + b_3 + b_4}{4}\right)$ or $\Re(\tilde{b}^{TzL}) = \left(\frac{b_1 + b_2 + b_3 + b_4}{4}\right)$

Division

$$\tilde{A}^{TzL} / \tilde{B}^{TzL} = \left(\frac{a_1}{\Re(\tilde{B}^{TzL})}, \frac{a_2}{\Re(\tilde{B}^{TzL})}, \frac{a_3}{\Re(\tilde{B}^{TzL})}, \frac{a_4}{\Re(\tilde{B}^{Tzl})}\right)$$

Where $\Re(\tilde{B}^{TzL}) = \left(\frac{b_1 + b_2 + b_3 + b_4}{4}\right)$ or $\Re(\tilde{b}^{TzL}) = \left(\frac{b_1 + b_2 + b_3 + b_4}{4}\right)$

Scalar multiplication

$$K\,\tilde{A}^{TzL} = \begin{cases} \left(ka_1, ka_2, ka_3, k\,a_4\right) & if\ K \geq 0 \\ \left(k\,a_4, ka_3, ka_2, ka_1\right) & if k < 0 \end{cases}$$

Definition 2.10 Trapezoidal fuzzy matrix (TrFM)

A trapezoidal fuzzy matrix of order m×n is defined as A $= (\tilde{a}_{ij}^{TzL})_{m\times n}$, where $\tilde{a}_{ij}^{TzL} = \left(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}\right)$ is the ij^{th} element of A.

Definition 2.11 Operations on Trapezoidal Fuzzy Matrices (TrFMs)

As for classical matrices. We define the following operations on trapezoidal fuzzy matrices. Let A = $(\tilde{a}_{ij}^{TzL})$ and B $=(\tilde{b}_{ij}^{TzL})$ be two trapezoidal fuzzy matrices (TrFMs) of same order. Then, we have the following

i. **Addition**

A+B = $\left(\tilde{a}_{ij}^{TzL} + \tilde{b}_{ij}^{TzL}\right)$

ii. **Subtraction**

A-B = $\left(\tilde{a}_{ij}^{TzL} - \tilde{b}_{ij}^{TzL}\right)$

iii. For A = $\left(\tilde{a}_{ij}^{TzL}\right)_{m\times n}$ and B = $\left(\tilde{b}_{ij}^{TzL}\right)_{n\times k}$ then AB = $\left(\tilde{c}_{ij}^{TzL}\right)_{m\times k}$ where $\tilde{c}_{ij}^{TzL} = \sum_{k=1}^{n} \tilde{a}_{ik}^{TzL} . \tilde{b}_{kj}^{TzL}$, i=1,2,…m and j=1,2,…p

iv. A^T or $A^1 = \left(\tilde{a}_{ji}^{TzL}\right)$

v. $KA = \left(K\,\tilde{a}_{ij}^{TzL}\right)$ where K is scalar.

Examples 2.12

1) If $A = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix}$ and B= $\begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$

Then A+B= $\left(\tilde{a}_{ij}^{TzL} + \tilde{b}_{ij}^{TzL}\right)$

$$A + B = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix} + \begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$$

$$A + B = \begin{bmatrix} (-3,0,6,9) & (-5,-1,11,15) \\ (-2,-3,8,13) & (-4,8,13,15) \end{bmatrix}$$

2) If $A = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix}$ and B= $\begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$

Then A-B= $\left(\tilde{a}_{ij}^{TzL} - \tilde{b}_{ij}^{TzL}\right)$

$$A - B = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix} - \begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$$

$$A - B = \begin{bmatrix} (-6,0,0,2) & (-9,-9,11,15) \\ (-9,4,7,6) & (-12,-8,-3,7) \end{bmatrix}$$

3) If $A = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix}$ and B= $\begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$

Then A.B= $(\tilde{a}_{ij}^{TzL}\tilde{b}_{ij}^{TzL})$

$$A.B = \begin{bmatrix} (-1,-2,3,4) & (-2,-4,6,8) \\ (-1,1,3,5) & (-2,1,4,5) \end{bmatrix} . \begin{bmatrix} (-2,2,3,5) & (-3,3,5,7) \\ (-1,-4,5,8) & (-2,7,9,10) \end{bmatrix}$$

$$A.B = \begin{bmatrix} (-1,-2,3,4)(2) + (-2,-4,6,8)(2) & (-1,-2,3,4)(3) + (-2,-4,6,8)(6) \\ (-1,1,3,5)(2) + (-2,1,4,5)(2) & (-1,1,3,5)(3) + (-2,1,4,5)(6) \end{bmatrix}$$

$$A.B = \begin{bmatrix} (-6,-12,18,24) & (-15,-30,45,60) \\ (-6,4,14,20) & (-15,9,33,45) \end{bmatrix}$$

Definition 2.13 Permanent

If $A = [a_{ij}]_{n\times n}$ is a crisp matrix of order $n \times n$, then the permanent of A is denoted by $Per(A)$ and defined as

$$Per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

Where S_n is the symmetric group of order n.
Let us consider an example to illustrate the permanent of a crisp matrix.

Example 2.14.

Let $A = \begin{bmatrix} 7 & 2 & 1 \\ 7 & 7 & 2 \\ 7 & 7 & 3 \end{bmatrix}$ be a crisp matrix of order 3×3.

Then $Per(A) = a_{11} . a_{22} . a_{33} + a_{12} . a_{23} . a_{31} + a_{13} . a_{22} . a_{31} + a_{11} . a_{23} . a_{32} +$
$a_{12} . a_{21} . a_{33} + a_{13} . a_{21} . a_{32}.$
$= 7 .7 .3 + 2 .2 .7 + 1 . 7 . 7 + 7 .2 .7 + 2 .7 .3 + 1 .7 .7 = 419.$

II. PERMANENT OF IVFMs

In this section, we introduce the permanent of IVFMs and some its properties.

Definition 3.1.

Let $\tilde{A} = [\tilde{a}_{ij}]_{m \times n}$ be an interval-valued fuzzy matrix, where $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$ be an interval and $0 \le a_{ijL} \le a_{ijU} \le 1$. Then the permanent of $\tilde{A}$ is denoted by $Per(\tilde{A})$ and defined by

$$Per(\tilde{A}) = \sum_{\sigma \in S} \prod_{i=1}^{m} a_{i\sigma(i)} \text{ for } m \le n$$

[where S is the set of all one-to-one mappings from $\{1,2, \dots n\} to \{1,2, \dots n\}$]

$$Per(\tilde{A}) = \sum_{\sigma \in S} \prod_{j=1}^{n} a_{\sigma(j)j} \text{ for } m > n$$

[where S is the set of all one-to-one mappings from $\{1,2, \dots n\}$ to $\{1,2, \dots m\}$]

Two expressions are written for the permanent of a matrix, because for $m > n$, there are one-to-one mappings from $\{1,2, \dots, n\} \; to \; \{1,2, \dots, m\}$. In this case, no one-to-one mapping is possible from $\{1,2, \dots, m\} \; to \; \{1,2, \dots, n\}$. But for $m \le n$ the one-to-one mappings are possible from $\{1,2, \dots, m\} \; to \; \{1,2, \dots, n\}$.

Following two examples are considered to illustrate the definition.

Example 3.2.

Let $\tilde{A} = \begin{bmatrix} [0.2,0.3] & [0.2,0.6] & [0.4,0.8] & [0.3,0.6] \\ [0.4,0.5] & [0.3,0.7] & [0.5,0.9] & [0.5,0.4] \end{bmatrix}$

Then, $per(\tilde{A}) = \max\{\min([0.2,0.3],[0.3,0.7]), \min([0.2,0.3],[0.5,0.9]),$
$\min([0.2,0.3],[0.5,0.4]), \min([0.2,0.6],[0.4,0.5]), \min([0.2,0.6],[0.5,0.9]),$
$\min([0.2,0.6],[0.5,0.4]), \min([0.4,0.8],[0.4,0.5]), \min([0.4,0.8],[0.3,0.7]),$
$\min([0.4,0.8],[0.5,0.4]), \min([0.3,0.6],[0.4,0.5]), \min([0.3,0.6],[0.3,0.7]),$
$\min([0.3,0.6],[0.5,0.9])\}$
$= \max\{[0.2,0.3],[0.2,0.3],[0.2,0.3],[0.2,0.5],[0.2,0.6],[0.2,0.4],[0.4,0.5],$
$[0.3,0.7],[0.4,0.4],[0.3,0.5],[0.3,0.6],[0.3,0.6]\}$
$= [0.4,0.7].$

Example 3.4.

If we take $\tilde{B} = \begin{bmatrix} [0.2,0.4] & [0.3,0.5] \\ [0.6,0.8] & [0.7,0.9] \\ [0.4,0.3] & [0.1,0.6] \\ [0.5,0.1] & [0.6,0.3] \end{bmatrix}$, then

$per(\tilde{B}) = \max\{\min([0.2,0.4],[0.7,0.9]), \min([0.2,0.4],[0.1,0.6]),$
$\min([0.2,0.4],[0.6,0.3]), \min([0.6,0.8],[0.3,0.5]), \min([0.6,0.8],[0.1,0.6]),$
$\min([0.6,0.8],[0.6,0.3]), \min([0.4,0.3],[0.3,0.5]), \min([0.4,0.3],[0.7,0.9]),$
$\min([0.4,0.3],[0.6,0.3]), \min([0.5,0.1],[0.3,0.5]), \min([0.5,0.1],[0.7,0.9]),$
$\min([0.5,0.1],[0.1,0.6])\}$
$= \max\{[0.2,0.4],[0.1,0.4],[0.2,0.3],[0.3,0.5],[0.1,0.6],[0.6,0.3],[0.3,0.3],$
$[0.4,0.3],[0.4,0.3],[0.3,0.1],[0.5,0.1],[0.1,0.1]\}$

$= [0.6,0.6]$.

2.4 Some properties of permanent of IVFMs.

Some trivial properties of permanent of IVFMs are presented below.

1. For any triangular or diagonal IVFM $\tilde{A}, Per(\tilde{A}) = \min\{of\ its\ diagonal\ entries\}$.
2. For any row IVFM or column IVFM $\tilde{A}, Per(\tilde{A}) = \max\{of\ the\ entries\}$.
3. For any two IVFMs $\tilde{A}\ and\ \tilde{B}$ such that, number of columns of $\tilde{A} = number\ of\ rows\ of\ \tilde{B}$, then $Per(\widetilde{AB}) \geq \min\{Per(\tilde{A}), Per(\tilde{B})\}$.
4. If $\tilde{A}$ and $\tilde{B}$ are any two IVFMs such that both $\widetilde{AB}$ and $\widetilde{BA}$ are defined, then $Per(\widetilde{AB}) \neq Per(\widetilde{BA})$, in general.

The proofs of the above properties are straightforward, they are illustrated by the following examples.

Example 3.5.

Let $\tilde{A} = \begin{bmatrix} [0.3,0.4] & [0.2,0.1] & [0.3,0.5] & [0.1,0.3] \\ [0.6,0.3] & [0.3,0.6] & [0.2,0.3] & [0.4,0.5] \end{bmatrix}$ and $\tilde{B} = \begin{bmatrix} [0.2,0.7] & [0.5,0.8] \\ [0.1,0.6] & [0.6,0.7] \\ [0.1,0.2] & [0.2,0.5] \\ [0.3,0.8] & [0.7,0.8] \end{bmatrix}$

Then $\widetilde{AB} = \begin{bmatrix} [0.1,0.2] & [0.2,0.4] & [0.3,0.5] \\ [0.3,0.7] & [0.3,0.6] & [0.4,0.2] \end{bmatrix}$

$Per(\tilde{A}) = \max\{[0.3,0.4], [0.2,0.1], [0.3,0.5], [0.1,0.3]\} = [0.3,0.5]$,

$Per(\tilde{B}) = \max\{[0.2,0.7], [0.1,0.6], [0.1,0.2], [0.3,0.8]\} = [0.3,0.8]$ and

$Per(\widetilde{AB}) = \max\{[0.1,0.2], [0.2,0.4], [0.3,0.5]\} = [0.3,0.5]$

Thus $Per(\widetilde{AB}) = \min\{Per(\tilde{A}), Per(\tilde{B})\}$.

In this example the equality part of the property 3 is satisfied.

Example 3.6. Let $\tilde{A} = [[0.1,0.5] \quad [0.5,0.9] \quad [0.2,0.3] \quad [0.2,0.6]]$ and $\tilde{B} = \begin{bmatrix} [0.2,0.3] \\ [0.3,0.4] \\ [0.2,0.6] \\ [0.2,0.5] \end{bmatrix}$

Then $\widetilde{AB} = [[0.3,0.5]]$, and $\widetilde{BA} = \begin{bmatrix} [0.2,0.3] & [0.2,0.4] & [0.2,0.5] & [0.2,0.6] \\ [0.1,0.6] & [0.1,0.5] & [0.1,0.4] & [0.1,0.3] \\ [0.3,0.2] & [0.3,0.1] & [0.3,0.7] & [0.3,0.4] \\ [0.2,0.1] & [0.2,0.7] & [0.3,0.5] & [0.1,0.7] \end{bmatrix}$

Now, $Per(\widetilde{BA}) = \max\{[0.2,0.3], [0.1,0.3], [0.3,0.1], [0.2,0.1]\} = [0.3,0.3]$.

For this case, $Per(\widetilde{AB}) \neq Per(\widetilde{BA})$. Which illustrate property 4.

In the following some important results are presented for permanent of IVFMs.

Proposition 3.7.

For any two IVFMs, $\tilde{A}\ and\ \tilde{B}$ of some order, such that $\tilde{A} \leq \tilde{B} \Rightarrow Per(\tilde{A}) \leq Per(\tilde{B})$.

Proof:

Let $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}\ and\ \tilde{B} = [\tilde{b}_{ij}]_{m\times n}$ be two IVFMs where $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$ $and\ \tilde{b}_{ij} = [b_{ijL}, b_{ijU}]$.

Then, $\tilde{A} \leq \tilde{B} \Rightarrow a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$ for all $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$. where $m \leq n$,

$$Per(\tilde{A}) = \sum_{\sigma\in S}\prod_{i=1}^{m}\tilde{a}_{i\sigma(i)} = \sum_{\sigma\in S}\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}]$$

$$\leq \sum_{\sigma\in S}\prod_{i=1}^{m}[b_{i\sigma(i)L}, b_{i\sigma(i)U}] = Per(\tilde{B}).$$

When $m > n$,

$$Per(\tilde{A}) = \sum_{\sigma\in S}\prod_{j=1}^{m}\tilde{a}_{\sigma(j)j} = \sum_{\sigma\in S}\prod_{j=1}^{m}[a_{\sigma(j)jL}, a_{\sigma(j)jU}]$$

$$\leq \sum_{\sigma\in S}\prod_{j=1}^{m}[b_{\sigma(j)jL}, b_{\sigma(j)jU}] = Per(\tilde{B}).$$

Hence $\tilde{A} \leq \tilde{B} \Rightarrow Per(\tilde{A}) \leq Per(\tilde{B})$.

Proposition 3.8.

For any IVFM $\tilde{A}, Per(\tilde{A}) = Per(\tilde{A}^T)$.

Proof:

Let $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ and $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$, when $m \le n$,

$$Per(\tilde{A}) = \sum_{\sigma\in S}\prod_{i=1}^{m}\tilde{a}_{i\sigma(i)} = \sum_{\sigma\in S}\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}]$$

Let $\tilde{A}^T = \tilde{B} = [\tilde{b}_{ij}]_{m\times n}, n \ge m.$

Then, $\tilde{b}_{ij} = \tilde{a}_{ij}$ i.e., $b_{ijL} = a_{ijL}$ *and* $b_{ijU} = a_{ijU}$.

$$Per(\tilde{A}^T) = Per(\tilde{B}) = \sum_{\sigma\in S}(\prod_{j=1}^{m} b_{\sigma(j)j}) = \sum_{\sigma\in S}\prod_{j=1}^{m} a_{j\sigma(j)}$$

$$= \sum_{\sigma\in S}\prod_{i=1}^{m} a_{i\sigma(i)} = Per(\tilde{A}).$$

For $m > n$, the proof is similar as brfore.

Proposition 3.9.

Interchanging of rows or column does not effect to the permanent value of the matrix.

Proof:

Let $\tilde{A} = [\tilde{a}_{ij}]$ be an IVFM of order $m \times n$ and $\tilde{B} = [\tilde{b}_{ij}]_{m\times n}$ is obtained from $\tilde{A}$ by interchanging the rth and sth row $(r < s)$ of $\tilde{A}$. Then it is clear that,

$\tilde{b}_{ij} = \tilde{a}_{ij}, i \ne r, i \ne s$ and $b_{rj} = a_{sj}, b_{sj} = a_{rj}$.

Now, $Per(\tilde{B}) = \sum_{\sigma\in S}(\prod_{i=1}^{m}\tilde{b}_{i\sigma(i)})$

$$= \sum_{\sigma\in S} b_{1\sigma(1)}b_{2\sigma(2)}\ldots b_{r\sigma(r)}\ldots b_{s\sigma(s)}\ldots b_{m\sigma(m)}$$

$$= \sum_{\sigma\in S} a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{s\sigma(r)}\ldots a_{r\sigma(s)}\ldots a_{m\sigma(m)}$$

$$= \sum_{\sigma\in S}[a_{1\sigma(1)L}, a_{1\sigma(1)U}][a_{2\sigma(2)L}, a_{2\sigma(2)U}]\ldots[a_{s\sigma(r)L}, a_{s\sigma(r)U}]\ldots[a_{r\sigma(s)L}, a_{r\sigma(s)U}]\ldots$$
$$[a_{m\sigma(m)L}, a_{m\sigma(m)U}].$$

Let $\lambda = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & s & \ldots & m \\ 1 & 2 & \ldots & s & \ldots & r & \ldots & m \end{pmatrix}$ and $\sigma\lambda = \phi, then\ \phi(i) = \sigma(i)$ for $i \ne r, i \ne s$ and $\phi(r) = \sigma(s) and\ \phi(s) = \sigma(r)$.

Then $Per(\tilde{B}) = \sum_{\sigma\in S}[a_{1\sigma(1)L}, a_{1\sigma(1)U}][a_{2\sigma(2)L}, a_{2\sigma(2)U}]\ldots[a_{s\sigma(r)L}, a_{s\sigma(r)U}]\ldots[a_{r\sigma(s)L}, a_{r\sigma(s)U}]\ldots$
$[a_{m\sigma(m)L}, a_{m\sigma(m)U}].$

$$= \sum_{\phi\in S}[a_{1\phi(1)L}, a_{1\phi(1)U}][a_{2\phi(2)L}, a_{2\phi(2)U}]\ldots[a_{s\phi(s)L}, a_{s\phi(s)U}]\ldots[a_{r\phi(r)L}, a_{r\phi(r)U}]\ldots$$
$$[a_{m\phi(m)L}, a_{m\phi(m)U}].$$

$$= \sum_{\phi\in S}[a_{1\phi(1)L}, a_{1\phi(1)U}][a_{2\phi(2)L}, a_{2\phi(2)U}]\ldots[a_{r\phi(r)L}, a_{r\phi(r)U}]\ldots[a_{s\phi(s)L}, a_{s\phi(s)U}]\ldots$$
$$[a_{m\phi(m)L}, a_{m\phi(m)U}].$$

$$= Per(\tilde{A}).$$

Hence, interchanging of rows or columns does not alter the permanent value.

A permutation matrix is a square binary matrix that has exactly one entry 1 in each row and each column and 0s elsewhere. An example of permutation matrix of order 3×3 is given by

$$\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It also can be written as $\tilde{A} = \begin{bmatrix} [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] \\ [0,0] & [1,1] & [0,0] \end{bmatrix}.$

Next proposition is related to permutation matrices.

Proposition 3.10

For any IVFM $\tilde{A}, Per(\tilde{A}) = Per(\tilde{P}\tilde{A}\tilde{Q})$, where $\tilde{P}$ and $\tilde{Q}$ are permutation matrices.

Proof:

Let $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$, $\tilde{P} = [\tilde{p}_{ij}]_{m\times m}$, and $\tilde{Q} = [\tilde{q}_{ij}]_{n\times n}$ where $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$, $\tilde{p}_{ij} = [p_{ijL}, p_{ijU}]$ and $\tilde{q}_{ij} = [q_{ijL}, q_{ijU}]$, where $\tilde{P}$ *and* $\tilde{Q}$ are permutation matrices.

The matrix $\tilde{A}\tilde{Q}$ obtained from $\tilde{A}$ by interchanging columns according as the matrix $\tilde{Q}$ [i.e. by which change we get $\tilde{Q}$ from I_n, we apply it to $\tilde{A}$].

Now $\tilde{P}\tilde{A}\tilde{Q} = \tilde{P}\tilde{A}_c$, where the IVFM $\tilde{P}\tilde{A}_c$ obtained from $\tilde{A}_c$ by interchanging rows according as the matrix $\tilde{P}$.

Let $\tilde{P}\tilde{A}_c = \tilde{A}_{cr}$, where $\tilde{A}_{cr}$ is the matrix obtained from $\tilde{A}$ by interchanging some rows and columns.

Now $Per(\tilde{P}\tilde{A}\tilde{Q}) = \mathrm{Per}(\tilde{A}_{cr}) = \mathrm{Per}(\tilde{A})$ [Since interchanging of rows and columns does not effect to the permanent value].

Let $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ *and* $\tilde{B} = [\tilde{b}_{ij}]_{m\times n}$ be two IVFMs where $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$ *and* $\tilde{b}_{ij} = [b_{ijL}, b_{ijU}]$.

Then we define $\tilde{a}_{ij} \wedge \tilde{b}_{ij} = [\min\{a_{ijL}, b_{ijL}\}, \min\{a_{ijU}, b_{ijU}\}]$,

$\tilde{a}_{ij} \vee \tilde{b}_{ij} = [\max\{a_{ijL}, b_{ijL}\}, \max\{a_{ijU}, b_{ijU}\}]$.

Also , $\tilde{A} \wedge \tilde{B} = [\tilde{a}_{ij} \wedge \tilde{b}_{ij}]_{m\times n}$, $\tilde{A} \vee \tilde{B} = [\tilde{a}_{ij} \vee \tilde{b}_{ij}]_{m\times n}$.

Proposition 3.11.

For any two IVFMs $\tilde{A}$ and $\tilde{B}$ of same order, $Per(\tilde{A} \wedge \tilde{B}) \le Per(\tilde{A}) \wedge Per(\tilde{B})$.

Proof:

Let $\tilde{A} \wedge \tilde{B} = \tilde{C}$, where $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$, $\tilde{B} = [\tilde{b}_{ij}]_{m\times n}$, $\tilde{C} = [\tilde{c}_{ij}]_{m\times n}$ are three IVFMs. Then $c_{ijL} = \min\{a_{ijL}, b_{ijL}\}$ and $c_{ijU} = \min\{a_{ijU}, b_{ijU}\}$. Now for $m \le n$,

$$Per(\tilde{A} \wedge \tilde{B}) = \sum_{\sigma\in S}\prod_{i=1}^{m} \tilde{c}_{i\sigma(i)} = \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[c_{i\sigma(i)L}, c_{i\sigma(i)U}]\right)$$

$$= \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[\min\{a_{i\sigma(i)L}, b_{i\sigma(i)L}\}, \min\{a_{i\sigma(i)U}, b_{i\sigma(i)U}\}]\right)$$

$$\le \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}]\right) = Per(\tilde{A}).$$

So, $Per(\tilde{A} \wedge \tilde{B}) \le Per(\tilde{A})$.

Now for, $m > n$, the proof is similar.

Similarly, we can prove $Per(\tilde{A} \wedge \tilde{B}) \le Per(\tilde{B})$.

Therefore, $Per(\tilde{A} \wedge \tilde{B}) \le Per(\tilde{A}) \wedge Per(\tilde{B})$.

Proposition 3.12.

For any two IVFMs $\tilde{A}$ and $\tilde{B}$ of same order, $Per(\tilde{A} \vee \tilde{B}) \ge Per(\tilde{A}) \vee Per(\tilde{B})$.

Proof:

Let $\tilde{A} \vee \tilde{B} = \tilde{D}$, where $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$, $\tilde{B} = [\tilde{b}_{ij}]_{m\times n}$, $\tilde{D} = [\tilde{d}_{ij}]_{m\times n}$ are three IVFMs. Then $d_{ijL} = \max\{a_{ijL}, b_{ijL}\}$ and $d_{ijU} = \max\{a_{ijU}, b_{ijU}\}$. Now for $m \le n$,

$$Per(\tilde{A} \vee \tilde{B}) = \sum_{\sigma\in S}\prod_{i=1}^{m} \tilde{d}_{i\sigma(i)} = \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[d_{i\sigma(i)L}, d_{i\sigma(i)U}]\right)$$

$$= \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[\max\{a_{i\sigma(i)L}, b_{i\sigma(i)L}\}, \max\{a_{i\sigma(i)U}, b_{i\sigma(i)U}\}]\right)$$

$$\ge \sum_{\sigma\in S}\left(\prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}]\right) = Per(\tilde{A}).$$

So, $Per(\tilde{A} \vee \tilde{B}) \ge Per(\tilde{A})$.

Now for, $m > n$, the proof is similar.

Similarly, we can prove $Per(\tilde{A} \vee \tilde{B}) \ge Per(\tilde{B})$.

Therefore, $Per(\tilde{A} \vee \tilde{B}) \geq Per(\tilde{A}) \vee Per(\tilde{B})$.
Hence the proof.

If $\tilde{A}$ *and* $\tilde{B}$ are two IVFMs satisfies the relation $\tilde{A}X\tilde{A} = \tilde{A}$, then X is called g-inverse of $\tilde{A}$ and $\tilde{A}$ is called regular.

Proposition 3.13.

If $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ be a regular matrix and $\tilde{B}$ is a g-inverse of $\tilde{A}$,
Then $Per(\tilde{A}\tilde{B}) = Per(\tilde{A}\tilde{B})^2$.

Proof:

Since $\tilde{B}$ is a g-inverse of $\tilde{A}$
Then $\tilde{A}\tilde{B}\tilde{A} = \tilde{A}$
$\Rightarrow \tilde{A}\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}$
$\Rightarrow (\tilde{A}\tilde{B})^2 = \tilde{A}\tilde{B}$
Then $Per(\tilde{A}\tilde{B}) = Per(\tilde{A}\tilde{B})^2$.
Hence the proof.

Proposition 3.14.

If $\tilde{A}$ is constant IVFM (i.e., all rows or all columns are equal) then

$$\begin{aligned} Per(\tilde{A}) &= \min\{its\ entries\} \\ &= \min\{along\ row\ entries\}\ if\ rows\ are\ equal \\ &= \min\{along\ column\ entires\}\ if\ columns\ are\ equal. \end{aligned}$$

Proof:

Let $\tilde{A} = [\tilde{a}_{ij}]_{m\times n}$ be a constant IVFM whose rows are equal.
Here $\tilde{a}_{ij} = [a_{ijL}, a_{ijU}]$ where $a_{1jL} = a_{ijL}$ and $a_{1jU} = a_{ijU}$ for all i. when $m > n$,

$$\begin{aligned} Per(\tilde{A}) &= \sum_{\sigma\in S}\prod_{j=1}^{m}[a_{\sigma(j)jL}, a_{\sigma(j)jU}] \\ &= \sum_{\sigma\in S}\prod_{j=1}^{m}[a_{1jL}, a_{1jU}] \\ &= \prod_{i=1}^{m}[a_{i\sigma(i)L}, a_{i\sigma(i)U}] \\ &= \min\{along\ row\ entries\}. \end{aligned}$$

When $m \leq n$, the proof is similar.
The proof is similar to order cases.

III. EVALUATION OF PERMANENT OF LARGE ORDER IVFM

It is very difficult to handle a large order matrix, it may be crisp or any kind of fuzzy matrix. Hence, evaluation of permanent of such matrices is also a complicated task. In this section, a suitable method is described to evaluate the permanent of an IVFM.

Let $\Omega_{pt} = \{(\omega_1, \omega_2, \dots, \omega_p): 1 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_p \leq t\}$; ω_i is integer, $i = 1,2,\dots,p$.
For $\alpha \in \Omega_{rm}, \beta \in \Omega_{sn}$, let $\tilde{A}[\alpha/\beta]$ denotes the $r \times s$ submatrixobtained from $\tilde{A}$ by taking only rows of α *abd columns of* β, where ijth entry of $\tilde{A}[\alpha/\beta]$ is $a_{\alpha_i\beta_j}$.
i.e., $\tilde{A}[\alpha/\beta]$ is the submatrix which we obtained from $\tilde{A}$ by deleting rows of α and columns of β.

Theorem 4.1:

For an $m \times n$ IVFM $\tilde{A}, m, n \geq 2$ *and* $\alpha \in \Omega_{rm}$, then

$$Per(\tilde{A}) = \sum_{\beta\in\Omega_{sn}}\left(\prod\{Per(\tilde{A}[\alpha/\beta]), Per(\tilde{A}(\alpha/\beta))\}\right).$$

In particular, $$Per(\tilde{A}) = \sum_{t=1}^{n}\left(\prod_{i=1}^{m}\{a_{it}, Per(\tilde{A}(i/t))\}\right).$$

Corollary 4.2:

Let $\tilde{A}$ be $m \times n$ matrix and $\tilde{A} = \begin{bmatrix}\tilde{B} & \tilde{C} \\ 0 & \tilde{D}\end{bmatrix}$ be a block presentation, where $\tilde{B} = [\tilde{b}_{ij}]_{m_1\times n_1}$,
$\tilde{C} = [\tilde{c}_{ij}]_{m_1\times n_2}, \tilde{D} = [\tilde{d}_{ij}]_{m_2\times n_2}$, such that $m_1 + m_2 = m,\ n_1 + n_2 = n$.

Then $Per(\tilde{A}) = Per(\tilde{B}) + Per(\tilde{D})$.

IV. PERMANENT OF TrFMs

In this section, we introduce the permanent of TrFM (i.e., Trapezoidal Fuzzy Number Matrices) and some of propositions.

Definition 5.1.

Let $\tilde{A}^{TzL} = [\tilde{a}_{ij}^{TzL}]_{m\times n}$ be a TrFM where $\tilde{a}_{ij}^{TzL} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number.
Then the permanent of $\tilde{A}^{TzL}$ is denoted by $Per(\tilde{A}^{TzL})$ and is defined by

$$Per(A^{TzL}) = \sum_{\sigma\in S}\prod_{i=1}^{m}(a_{1i\sigma(i)}, a_{2i\sigma(i)}, a_{3i\sigma(i)}, a_{4i\sigma(i)}) \text{ for } m \le n$$

[where S is the set of all one-to-one mapping from $\{1,2,\dots,m\}$ to $\{1,2,\dots,n\}$]

$$= \sum_{\sigma\in S}\prod_{j=1}^{n}(a_{1\sigma(j)j}, a_{2\sigma(j)j}, a_{3\sigma(j)j}, a_{4\sigma(j)j}) \text{ for } mn$$

[where S is the set of all one-to-one mapping from $\{1,2,\dots,n\}$ to $\{1,2,\dots,m\}$]

5.2. Permanent of some special types of TrFMs.

Here we deduce the permanent of some special types of TrFMs.

(1) **Pure Null TrFM:**
If $\tilde{A}^{TzL}$ is pure null TrFM then (i.e. all entries are zero), then $Per(\tilde{A}^{TzL}) = (0,0,0,0)$.

(2) **Fuzzy Null TrFM:**
If $\tilde{A}^{TzL}$ is a fuzzy null TrFM (i.e. all $\tilde{a}_{ij}^{TzL} = (0, \epsilon_1, \epsilon_2, \epsilon_3)$for all i, j where $\epsilon_1\epsilon_2\epsilon_3 \neq 0$), then $Per(\tilde{A}) = (0,0,0,0)$.

(3) **Pure Unit TrFM:**
If $\tilde{A}^{TzL}$ is a pure unit TrFM (i.e. a square TrFM where $\tilde{a}_{ii}^{TzL} = (1,0,0,0)$ and $\tilde{a}_{ij}^{TzL} = (0,0,0,0)$ for all $i \neq j$), then $Per(\tilde{A}^{TzL}) = (1,0,0,0)$.

(4) **Fuzzy Unit TrFM:**
If $\tilde{A}^{TzL}$ is a fuzzy unit TrFM (i.e. a square TrFM where $\tilde{a}_{ii}^{TzL} = (1, \epsilon_1, \epsilon_2, \epsilon_3)$ and $\tilde{a}_{ij}^{TzL} = (0, \epsilon_4, \epsilon_5, \epsilon_6)$ for all $i \neq j$ where $\epsilon_1\epsilon_2\epsilon_3 \neq 0, \epsilon_4\epsilon_5\epsilon_6 \neq 0$), then $Per(\tilde{A}^{TzL}) = \prod \tilde{a}_{ii}^{TzL}$.

(5) **Pure Triangular TrFM:**
If $\tilde{A}^{TzL}$ is a pure triangular TrFM (i.e. a square TrFM where either $\tilde{a}_{ij}^{TzL} = (0,0,0,0)$ for all $i > j$ or $\tilde{a}_{ij}^{TzL} = (0,0,0,0)$ for all $i < j$), then $Per(\tilde{A}^{TzL}) = \prod \tilde{a}_{ii}^{TzL}$.

(6) **Fuzzy Triangular TrFM:**
If $\tilde{A}^{TzL}$ is a fuzzy triangular TrFM (i.e. a square TrFM where either $\tilde{a}_{ij}^{TzL} = (0, \epsilon_1, \epsilon_2, \epsilon_3)$ for all $i > j$ or $\tilde{a}_{ij}^{TzL} = (0, \epsilon_1, \epsilon_2, \epsilon_3)$ for all $i < j$ and $\epsilon_1\epsilon_2\epsilon_3 \neq 0$), then $Per(\tilde{A}^{TzL}) = \prod \tilde{a}_{ii}^{TzL}$.

Here some proposition related to the permanent of TrFM are given.

Proposition 5.3.

Let $\tilde{A}^{TzL}$ be a TrFM of order $m \times n$. If all the elements of a row(column) of $\tilde{A}^{TzL}$ are (0,0,0,0), then $Per(\tilde{A}^{TzL}) = (0,0,0,0)$.

Proof:

Let $\tilde{A}^{TzL} = [\tilde{a}_{ij}^{TzL}]_{m\times n}$ be a TrFM of order $\times\, n$, where $\tilde{a}_{ij}^{TzL} = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij})$.
Let $m \le n$, then

$$Per(\tilde{A}^{TzL}) = \sum_{\sigma\in S}(a_{11\sigma(1)}, a_{21\sigma(1)}, a_{31\sigma(1)}, a_{41\sigma(1)})(a_{12\sigma(2)}, a_{22\sigma(2)}, a_{32\sigma(2)}, a_{42\sigma(2)})(a_{13\sigma(3)}, a_{23\sigma(3)}, a_{33\sigma(3)}, a_{43\sigma(3)})\dots$$
$$(a_{1m\sigma(m)}, a_{2m\sigma(m)}, a_{3m\sigma(m)}, a_{4m\sigma(m)})$$

Here $Per(\tilde{A}^{TzL})$ is the sum of n_{P_m} expressions, where in each expression there is a term (0,0,0,0).
So each expression becomes (0,0,0,0) and the sum is (0,0,0,0,).
$\therefore\ Per(\tilde{A}^{TzL}) = (0,0,0,0)$
For $m > n$, the proof is similar.

Proposition 5.4.

Let $\tilde{A}^{TzL}$ be a TrFM of order $m \times n$. If a row be multiplied by a scalar K then the permanent value will be $kPer(\tilde{A}^{TzL})$.

Proof:

Let $\tilde{A}^{TzL} = [\tilde{a}_{ij}^{TzL}]_{m\times n}$ be a TrFM of order $m \times n$ and $\tilde{B}^{TzL} = [\tilde{b}_{ij}^{TzL}]_{m\times n}$ be another TrFM obtained by multiplying k to a row of $\tilde{A}^{TzL}$.

Let $\tilde{a}_{ij}^{TzL} = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij})$, $\tilde{b}_{ij}^{TzL} = (b_{1ij}, b_{2ij}, b_{3ij}, b_{4ij})$ and k is multiplied to the rth row.

Let $m \le n$, then

$$Per(\tilde{B}^{TzL}) = \sum_{\sigma\in S}(b_{11\sigma(1)}, b_{21\sigma(1)}, b_{31\sigma(1)}, b_{41\sigma(1)})(b_{12\sigma(2)}, b_{22\sigma(2)}, b_{32\sigma(2)}, b_{42\sigma(2)})(b_{13\sigma(3)}, b_{23\sigma(3)}, b_{33\sigma(3)}, b_{43\sigma(3)})\ldots$$
$$(b_{1r\sigma(r)}, b_{2r\sigma(r)}, b_{3r\sigma(r)}, b_{4r\sigma(r)})\ldots(b_{1m\sigma(m)}, b_{2m\sigma(m)}, b_{3m\sigma(m)}, b_{4m\sigma(m)}).$$

$$= \sum_{\sigma\in S}(a_{11\sigma(1)}, a_{21\sigma(1)}, a_{31\sigma(1)}, a_{41\sigma(1)})(a_{12\sigma(2)}, a_{22\sigma(2)}, a_{32\sigma(2)}, a_{42\sigma(2)})(a_{13\sigma(3)}, a_{23\sigma(3)}, a_{33\sigma(3)}, a_{43\sigma(3)})\ldots$$
$$k(a_{1r\sigma(r)}, a_{2r\sigma(r)}, a_{3r\sigma(r)}, a_{4r\sigma(r)})\ldots(a_{1m\sigma(m)}, a_{2m\sigma(m)}, a_{3m\sigma(m)}, a_{4m\sigma(m)}).$$
$$= k\sum_{\sigma\in S}(a_{11\sigma(1)}, a_{21\sigma(1)}, a_{31\sigma(1)}, a_{41\sigma(1)})(a_{12\sigma(2)}, a_{22\sigma(2)}, a_{32\sigma(2)}, a_{42\sigma(2)})(a_{13\sigma(3)}, a_{23\sigma(3)}, a_{33\sigma(3)}, a_{43\sigma(3)})\ldots$$
$$(a_{1r\sigma(r)}, a_{2r\sigma(r)}, a_{3r\sigma(r)}, a_{4r\sigma(r)})\ldots(a_{1m\sigma(m)}, a_{2m\sigma(m)}, a_{3m\sigma(m)}, a_{4m\sigma(m)}).$$
$$= kPer(\tilde{A}^{TzL}).$$

Proof is similar for $m > n$.

Proposition 5.5.

If any two rows (or columns) of a TrFM $\tilde{A}^{TzL}$ are interchanged, then permanent value remains unchanged.

Proof:

Let $\tilde{A}^{TzL} = [\tilde{a}_{ij}^{TzL}]_{m\times n}$ be a TrFM of order $m \times n$ and $\tilde{B}^{TzL} = [\tilde{b}_{ij}^{TzL}]_{m\times n}$ is the TrFM obtained from $\tilde{A}^{TzL}$ by interchanging the rth and sth row $(r < s)$ of $\tilde{A}^{TzL}$.

Then $\tilde{b}_{ij}^{TzL} = \tilde{a}_{ij}^{TzL}, i \ne r, j \ne s$ *and* $\tilde{b}_{rj}^{TzL} = \tilde{a}_{sj}^{TzL}$ and $\tilde{b}_{sj}^{TzL} = \tilde{a}_{rj}^{TzL}$.

Let $m \le n$. Now,

$$Per(\tilde{B}^{TzL}) = \sum_{\sigma\in S} b_{1\sigma(1)}b_{2\sigma(2)}\cdots b_{r\sigma(r)}\cdots b_{s\sigma(s)}\cdots b_{m\sigma(m)}$$
$$= \sum_{\sigma\in S} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{s\sigma(s)}\cdots a_{r\sigma(r)}\cdots a_{m\sigma(m)}$$
$$= \sum_{\sigma\in S}(a_{11\sigma(1)}, a_{21\sigma(1)}, a_{31\sigma(1)}, a_{41\sigma(1)})(a_{12\sigma(2)}, a_{22\sigma(2)}, a_{32\sigma(2)}, a_{42\sigma(2)})\ldots$$
$$(a_{1s\sigma(s)}, a_{2s\sigma(s)}, a_{3s\sigma(s)}, a_{4s\sigma(s)})\ldots(a_{1r\sigma(r)}, a_{2r\sigma(r)}, a_{3r\sigma(r)}, a_{42r(r)})\ldots$$
$$(a_{1m\sigma(m)}, a_{2m\sigma(m)}, a_{3m\sigma(m)}, a_{4m\sigma(m)}).$$
$$= \sum_{\sigma\in S}(a_{11\sigma(1)}, a_{21\sigma(1)}, a_{31\sigma(1)}, a_{41\sigma(1)})(a_{12\sigma(2)}, a_{22\sigma(2)}, a_{32\sigma(2)}, a_{42\sigma(2)})\ldots$$
$$(a_{1r\sigma(r)}, a_{2r\sigma(r)}, a_{3r\sigma(r)}, a_{42r(r)})\ldots(a_{1s\sigma(s)}, a_{2s\sigma(s)}, a_{3s\sigma(s)}, a_{4s\sigma(s)})\ldots$$
$$(a_{1m\sigma(m)}, a_{2m\sigma(m)}, a_{3m\sigma(m)}, a_{4m\sigma(m)}).$$
$$= Per(\tilde{A}^{TzL}).$$

For $m > n$, the proof is similar as before.

Proposition 5.6:

If $\tilde{A}^{Tzl^T}$ is the transpose of $\tilde{A}^{TzL}$, then $Per\left(\tilde{A}^{Tzl^T}\right) = Per(\tilde{A}^{TzL})$.

Proof:

Let $\tilde{A}^{TzL} = [\tilde{a}_{ij}^{TzL}]_{m\times n}$ be a TrFM and let $\tilde{A}^{TzL^T} = \tilde{B}^{TzL} = [\tilde{b}_{ij}^{TzL}]_{m\times n}$ when $\tilde{b}_{ij}^{TzL} = \tilde{a}_{ij}^{TzL}$.

When $m > n$

$$Per(\tilde{A}^{TzL^T}) = Per(\tilde{B}^{TzL}) = \sum_{\sigma\in S} b_{1\sigma(1)}b_{2\sigma(2)}\ldots b_{n\sigma(n)}$$
$$= \sum_{\sigma\in S} a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}$$

$$= \sum_{\sigma \in S} \left(a_{1\sigma(1)1}, a_{2\sigma(1)1}, a_{3\sigma(1)1}, a_{4\sigma(1)1}\right)\left(a_{1\sigma(2)2}, a_{2\sigma(2)2}, a_{3\sigma(2)2}, a_{4\sigma(2)2}\right)\ldots$$
$$\left(a_{1\sigma(n)n}, a_{2\sigma(n)n}, a_{3\sigma(n)n}, a_{4\sigma(n)n}\right).$$
$$= \sum_{\sigma \in S} \prod_{i=1}^{n} (a_{1\sigma(j)j}, a_{2\sigma(j)j}, a_{3\sigma(j)j}, a_{4\sigma(j)j})$$
$$= Per\left(\tilde{A}^{TzL}\right).$$

For $m \leq n$, the proof is similar.

6 CONCLUSION

Permanent of both IVFM and TrFM are defined where the matrices are not necessary to be required.Related properties are given with proper examples. Some properties are stated and proves also. Infuture we should try to develop of the permanent of block fuzzy matrix as well as block IVFM.

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