Associated properties of β-$\theta_g\beta$-closed functions

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ABSTRACT: In this paper, a new class of sets called $\theta_g\beta$-closed set was introduced by [16]. Using this sets, a new class of functions called pre-$\theta_g\beta$-closed functions is introduced. Properties of these concepts are widely discussed in this paper.

Keywords: β-open set, β-clopen set, gβ-closed set, semi pre-closed set, $\theta_g\beta$-closed set, θ-closed set.

1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc., by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of β-open [1] sets introduced by Abd El-Monsef in 1983.

In 1970, Levine [7] defined and studied generalized closed sets in topological spaces. In 1982, Malghan [10] defined generalized closed functions and obtained some preservation theorems of normality and regularity. In 1990, Arya and Nour [5] defined generalized semi-open sets and used them to obtain characterizations of s-normal spaces due to Maheshwari and Prasad [8]. In 1993, Devi et. al. [6] defined and studied generalized semi-closed functions and showed that the continuous generalized semi-closed surjective image of a normal space is s-normal. In 1998, Noiri et.al. [13] defined generalized pre-closed sets and introduced generalized pre-closed functions and showed that the continuous generalized pre-closed surjective image of normal space is prenormal [15] (or p-normal [14]). Recently, Tahiliani [17] has defined generalized β-closed functions and has shown that the continuous generalized β-closed surjective images of normal (resp. regular) spaces are β-normal [9] (resp. β-regular [3]). Further, it has shown that β-regularity is preserved under continuous pre-β-open [9] β-gβ-closed surjections. Recently, Tahiliani [16] has defined $\theta_g\beta$-closed sets and studied properties and characterizations of them.

A subset A of X is said to be β-open [1] (= semi pre-open [4]) if $A \subseteq cl(int(cl(A)))$.

The complement of β-open (resp. regular open) set is called β-closed (= semi pre-closed) (resp. regular closed).

The intersection of all β-closed sets of X containing A is called the β-closure [2] (= semi pre-closure) of A and is denoted by βcl(A) (= spcl(A)).

It is evident that a set A is β-closed if and only if βcl(A) = A [4].

The β-interior [2] of A, βint(A), is the union of all β-open sets contained in A.

A subset A of X is said to be β-clopen [11] or semi-pre-regular [12] or β-regular if it is β-open and β-closed.

The family of all β-open (resp. β-closed, β-clopen, regular closed) sets of X is denoted by βO(X) (resp. βC(X), βR(X), RC(X)).

A subset A of a topological space (X, τ) is called a generalized β-closed [17] (briefly gβ-closed) set of X if βcl(A) ⊆ U holds whenever A ⊆U and U is open in X.

A will be called gβ-open if X\A is gβ-closed.

Remark 1.1 [1] Every open set is β-open but not conversely.

Remark 1.2 [17] Every β-open set is gβ-open but not conversely.

Theorem 1.3 [4] For any subset A of a topological space X, the following conditions are equivalent:

1. A ∈ βO(X);
2. A ⊆ cl(int(cl(A)));
3. cl(A) ∈ RC(X).

2. Pre-$\theta_g\beta$-closed functions

Definition 2.1 A subset A of a topological space (X, τ) is called $\theta_g\beta$-closed set of X if βcl(A) ⊆ U holds whenever A ⊆ U and U is θ-open in X.

A will be called $\theta_g\beta$-open if X\A is $\theta_g\beta$-closed.

Remark 2.2 [16] Every gβ-open set is $\theta_g\beta$-open but not conversely.

Definition 2.3 A function f : X → Y is said to be pre-$\theta_g\beta$-closed (= β-$\theta_g\beta$-closed) (resp. regular $\theta_g\beta$-closed, almost $\theta_g\beta$-closed) if for each F ∈ βC(X) (resp. F ∈ βR(X), F ∈ RC(X)), f(F) is $\theta_g\beta$-closed in Y.

Definition 2.4 A function f : X → Y is said to be $\theta_g\beta$-closed if for each closed set F of X, f(F) is $\theta_g\beta$-closed in Y.

From the above definitions, we obtain the following diagram:

$$\begin{array}{ccc} \text{pre-}\theta_g\beta\text{-closed} & \rightarrow & \text{regular } \theta_g\beta\text{-closed} \\ \downarrow & & \downarrow \\ \theta_g\beta\text{-closed} & \rightarrow & \text{almost } \theta_g\beta\text{-closed} \end{array}$$

Remark 2.5 None of all implications in the above diagram is reversible as the following examples show.

Example 2.6 Let X = Y = {a, b, c, d}, τ = {θ, X, {b}, {b, d}, {a, b, c}} and σ = {θ, Y, {c}, {d}, {b, c}, {c, d}, {a, c, d}, {b, c, d}}. Let f : (X, τ) → (Y, σ) be the identity function. Then f is both regular $\theta_g\beta$-closed and $\theta_g\beta$-closed but it is not pre-$\theta_g\beta$-closed.

Example 2.7 Let X = Y = {a, b, c}, τ = {θ, X, {a}, {a, d}, {a, b, d}, {a, c, d}} and σ = {θ, Y, {c}, {d}, {b, c}, {c, d}, {a, c, d}, {b, c, d}}. Let f : (X, τ) → (Y, σ) be the identity function. Then f is almost $\theta_g\beta$-closed but not $\theta_g\beta$-closed.

Example 2.8 Let X = Y = {a, b, c, d}, τ = {θ, X, {b, c}, {a, b, c}, {b, c, d}} and σ = {θ, Y, {c}, {d}, {b, c}, {c, d}, {a, c, d}, {b, c, d}}. Let f : (X, τ) → (Y, σ) be the identity function. Then f is almost $\theta_g\beta$-closed but not regular $\theta_g\beta$-closed.

The proof of the following Lemma follows using a standard technique and thus omitted. **Lemma 2.9** A surjective function f : X → Y is pre-$\theta_g\beta$-closed (resp. regular $\theta_g\beta$-closed) if and only if for each subset B of Y and each U ∈ βO(X) (resp. U ∈ βR(X)) containing $f^{-1}(B)$, there exists a $\theta_g\beta$-open set V of Y such that B ⊆ V and $f^{-1}(V) \subseteq U$.

Corollary 2.10 If a surjective function f : X → Y is pre-$\theta_g\beta$-closed (resp. regular $\theta_g\beta$-closed), then for each θ-closed set K of Y and each U ∈ βO(X) (resp. U ∈ βR(X)) containing $f^{-1}(K)$, there exists V ∈ βO(Y) containing K such that $f^{-1}(V) \subseteq U$.

Proof. Suppose that f: X →Y is pre-$\theta_g\beta$-closed (resp. regular $\theta_g\beta$-closed). Let K be any θ-closed set of Y and U ∈ βO(X) (resp. U ∈ βR(X)) containing $f^{-1}(K)$. By Lemma 2.9, there exists a $\theta_g\beta$-open set G of Y such that K ⊆G and $f^{-1}(G) \subseteq U$. Since K is θ-closed, by Lemma 2.9, K ⊆ βint(G). Put V = βint(G). Then, K ⊆V ∈ βO(Y) and $f^{-1}(V) \subseteq U$.

Definition 2.11 A function f : X → Y is said to be

1. θ-irresolute if $f^{-1}(F)$ is θ-closed in X for every θ-closed set F of Y.
2. m-θ-closed if f(F) is θ-closed in Y for every θ-closed set F of X.

Lemma 2.12 A function f : X → Y is θ-irresolute if and only if $f^{-1}(F)$ is θ-open in X for every θ-open set F of Y.

Theorem 2.13 If f : X → Y is θ-irresolute pre-$\theta_g\beta$-closed bijection, then f(H) is $\theta_g\beta$-closed in Y for each $\theta_g\beta$-closed set H of X.

Proof. Let H be any $\theta_g\beta$-closed set of X and V an θ-open set of Y containing f(H). Since $f^{-1}(V)$ is an θ-open set of X containing H, βcl(H) ⊆ $f^{-1}(V)$ and hence f(βcl(H)) ⊆ V. Since f is pre-$\theta_g\beta$-closed and βcl(H) ∈ βC(X), f(βcl(H)) is $\theta_g\beta$-closed in Y. We have βcl(f(H)) ⊆ βcl(f(βcl(H))) ⊆ V. Therefore, f(H) is $\theta_g\beta$-closed in Y.

Definition 2.14 A function f : X → Y is said to be pre-$\theta_g\beta$-continuous or β-$\theta_g\beta$-continuous if $f^{-1}(K)$ is $\theta_g\beta$-closed in X for every K ∈ βC(Y).

It is obvious that a function f: X →Y is pre-$\theta_g\beta$-continuous if and only if $f^{-1}(V)$ is $\theta_g\beta$-open in X for every V ∈ βO(Y).

Theorem 2.15 If f : X → Y is m-θ-closed pre-$\theta_g\beta$-continuous bijection, then $f^{-1}(K)$ is $\theta_g\beta$-closed in X for each $\theta_g\beta$-closed set K of Y.

Proof. Let K be $\theta_g\beta$-closed set of Y and U an θ-open set of X containing $f^{-1}(K)$. Put V = Y −f(X −U), then V is an θ-open in Y, K ⊆ V and $f^{-1}(V) \subseteq U$. Therefore, we have βcl(K) ⊆ V and hence $f^{-1}(K) \subseteq f^{-1}(\beta cl(K)) \subseteq f^{-1}(V) \subseteq U$. Since f is pre-$\theta_g\beta$-continuous and βcl(K) is β-closed in Y, $f^{-1}(\beta cl(K))$ is $\theta_g\beta$-closed in X and hence $\beta cl(f^{-1}(K)) \subseteq \beta cl(f^{-1}(\beta cl(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $\theta_g\beta$-closed in X.

Recall that a function f: X →Y is said to be β-irresolute [26] if $f^{-1}(V) \in \beta O(X)$ for every $V \in \beta O(Y)$.

Remark 2.16 Every β-irresolute function is pre-$\theta_g\beta$-continuous but not conversely.

Proof. Let $A \in \beta O(Y)$. Since f is β-irresolute, $f^{-1}(A) \in \beta O(X)$ and so, by Remarks 1.2 and 2.2, $f^{-1}(A)$ is $\theta_g\beta$-open in X. Hence f is pre-$\theta_g\beta$-continuous.

Example 2.17 Let X = Y = {a, b, c, d}, τ = {θ, X, {b, d}, {a, b, d}} and σ = {θ, Y, {a}, {b}, {a, b}, {b, c}, {a, b, c}}. Let f : (X, τ) → (Y, σ) be the identity function. Then f is pre-$\theta_g\beta$-continuous but not β-irresolute.

Corollary 2.18 If f : X → Y is m-θ-closed β-irresolute bijection, then $f^{-1}(K)$ is $\theta_g\beta$-closed in X for each $\theta_g\beta$-closed set K of Y.

Proof. It is obtained from Theorem 2.15**.**

For the composition of pre-$\theta_g\beta$-closed functions, we have the following Theorems.

Theorem 2.19 Let f : X → Y and g : Y → Z be functions. Then the composition gof : X → Z is pre-$\theta_g\beta$-closed if f is pre-$\theta_g\beta$-closed and g is θ- irresolute pre-$\theta_g\beta$-closed bijection.

Proof. The proof follows immediately from Theorem 2.13.

Theorem 2.20 Let f : X → Y and g : Y → Z be functions and let the composition gof : X → Z be pre-$\theta_g\beta$-closed. Then the following hold:

1. If f is a β-irresolute surjection, then g is pre-$\theta_g\beta$-closed;
2. If g is a m-θ-closed pre-$\theta_g\beta$-continuous injection, then f is pre-$\theta_g\beta$-closed.

Proof. (1) Let $K \in \beta C(Y)$. Since f is β-irresolute and surjective, $f^{-1}(K) \in \beta C(X)$ and $(gof)(f^{-1}(K)) = g(K)$. Therefore, g(K) is $\theta_g\beta$-closed in Z and hence g is pre-$\theta_g\beta$-closed.

(2) Let $H \in \beta C(X)$. Then (gof)(H) is $\theta_g\beta$-closed in Z and $g^{-1}((gof)(H)) = f(H)$. By Theorem 2.15, f(H) is $\theta_g\beta$-closed in Y and hence f is pre-$\theta_g\beta$-closed.

The following Lemma is analogous to Lemma 2.9, the straightforward proof is omitted.

Lemma 2.21 A surjective function f : X → Y is almost $\theta_g\beta$-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a $\theta_g\beta$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Corollary 2.22 If a surjective function f : X → Y is almost $\theta_g\beta$-closed, then for each θ-closed set K of Y and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in \beta O(Y)$ such that $K \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. The proof is similar to that of Corollary 2.10.

A topological space (X, τ) is said to be θ-normal if for every disjoint θ-closed sets A and B of X, there exist disjoint sets U, V ∈ τ such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.23 A topological space (X, τ) is said to be θ-β-normal if for every disjoint θ-closed sets A and B of X, there exist disjoint sets U, V ∈ βO(X) such that $A \subseteq U$ and $B \subseteq V$.

Theorem 2.24 Let f : X → Y be a θ-irresolute almost $\theta_g\beta$-closed surjection. If X is θ-normal, then Y is θ-β-normal.

Proof. Let K_1 and K_2 be any disjoint θ-closed sets of Y. Since f is θ-irresolute, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint θ-closed sets of X. By the θ-normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(K_i) \subseteq U_i$, where i = 1,2. Now, put $G_i = int(cl(U_i))$ for i = 1, 2, then $G_i \in RO(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \theta$. By Corollary 2.22, there exists $V_i \in \beta O(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, i= 1, 2. Since $G_1 \cap G_2 = \theta$, f is surjective we have $V_1 \cap V_2 = \theta$. This shows that Y is θ-β-normal.

Definition 2.25 [1] A function $f : X \to Y$ is said to be β-open (resp. β-closed), if $f(U) \in \beta O(Y)$ (resp. $f(U) \in \beta C(Y)$) for every open (resp. closed) set U of X.

Definition 2.26 [17] A function $f : X \to Y$ is said to be gβ-closed if f(U) is gβ-closed in Y for every closed set U of X.

The following four Corollaries are immediate consequences of Theorem 2.24.

Corollary 2.27 If $f : X \to Y$ is a θ-irresolute $\theta_g\beta$-closed surjection and X is θ-normal, then Y is θ-β-normal.

Corollary 2.28 If $f : X \to Y$ is a θ-irresolute gβ-closed surjection and X is θ-normal, then Y is θ-β-normal.

Corollary 2.29 If $f : X \to Y$ is a θ-irresolute β-closed surjection and X is θ-normal, then Y is θ-β-normal.

Corollary 2.30 If $f : X \to Y$ is a θ-irresolute closed surjection and X is θ-normal, then Y is θ-β-normal.

Definition 2.31 [9] A function $f : X\to Y$ is said to be pre-β-closed (resp. pre-β-open) if for each $F \in \beta C(X)$ (resp. $F \in \beta O(X)$), $f(F) \in \beta C(Y)$ (resp. $f(F) \in \beta O(Y)$).

Remark 2.32 Every pre-β-closed function is β-closed but not conversely.

Proof. Let A be a closed set of X. Then A is β-closed set of X. Since f is pre-β-closed, $f(A) \in \beta C(Y)$. Hence f is β-closed.

Example 2.33 Let $X = Y = \{a, b, c\}$, $\tau = \{\theta, X, \{a, b\}\}$ and $\sigma = \{\theta, Y, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is β-closed but not pre-β-closed.

Theorem 2.34 [12] Let A be a subset of a topological space X. Then

1. $A \in \beta O(X)$ if and only if $\beta cl(A) \in \beta R(X)$.
2. $A \in \beta C(X)$ if and only if $\beta int(A) \in \beta R(X)$.

Theorem 2.35 Let $f : X \to Y$ be a θ-irresolute regular $\theta_g\beta$-closed surjection. If X is θ-β-normal, then Y is θ-β-normal.

Proof. Although the proof is similar to that of Theorem 2.24, we will state it for the convenience of the reader. Let K_1 and K_2 be any disjoint θ-closed sets of Y. Since f is θ-irresolute, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint

θ-closed sets of X. By the θ-β-normality of X, there exist disjoint sets U1, U2 $\in \beta O(X)$ such that $f^{-1}(K_i) \subseteq$ Ui, for I = 1,2. Now, put $G_i = \beta cl(Ui)$ for i=1, 2, then by Theorem2.34, $G_i \in \beta R(X)$, $f^{-1}(K_i) \subseteq U_i \subseteq G_i$ and $G_1 \cap G_2 = \theta$. By Corollary 2.10, there exists $V_i \in \beta O(Y)$ such that $K_i \subseteq V_i$ and $f^{-1}(V_i) \subseteq G_i$, where i= 1, 2. Since f is surjective and $G_1 \cap G_2 = \theta$, we obtain $V_1 \cap V_2 = \theta$. This shows that Y is θ-β-normal.

Corollary 2.36 Let $f : X \to Y$ be a θ-irresolute pre-$\theta_g\beta$-closed surjection. If X is θ-β-normal, then Y is θ-β-normal.

Remark 2.37 Every pre-β-closed function is pre-$\theta_g\beta$-closed but not conversely.

Proof. Let $F \in \beta C(X)$. Since f is pre-β-closed, $f(F) \in \beta C(Y)$ and so, by Remarks 1.2 and 2.2, f(F) is $\theta_g\beta$-closed in Y. Hence f is pre-$\theta_g\beta$-closed.

Example 2.38 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\theta, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\sigma = \{\theta, Y, \{b, d\}, \{a, b, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is pre-$\theta_g\beta$-closed but not pre-β-closed.

Corollary 2.39 If $f : X \to Y$ is a θ-irresolute pre-β-closed surjection and X is θ-β-normal, then Y is θ-β-normal.

Theorem 2.40 Let f : X → Y be a m-θ-closed pre-$\theta_g\beta$-continuous injection. If Y is θ-β-normal, then X is θ-β-normal.

Proof. Let H_1 and H_2 be any disjoint θ-closed sets of X. Since f is a m-θ-closed injection, $f(H_1)$ and $f(H_2)$ are disjoint θ-closed sets of Y. By the θ-β-normality of Y, there exist disjoint sets $V_1, V_2 \in \beta O(Y)$ such that $f(H_i) \subseteq V_i$, for i=1,2. Since f is pre-$\theta_g\beta$-continuous $f^{-1}(V_i)$ and $f^{-1}(V_i)$ are disjoint $\theta_g\beta$-open sets of X and $H_i \subseteq f^{-1}(V_i)$ for i=1,2. Now, put $U_i = \beta int(f^{-1}(V_i))$ for i=1,2. Then $U_i \in \beta O(X)$, $H_i \subseteq U_i$ and $U_1 \cap U_2 = \theta$. This shows that X is θ-β-normal. **Corollary 2.41** If f : X → Y is a m-θ-closed β-irresolute injection and Y is θ-β-normal, then X is θ-β-normal.

Proof. This is an immediate consequence of Theorem 2.40, since every β-irresolute function is pre-$\theta_g\beta$-continuous.

Definition 2.42 A topological space X is said to be θ-regular if for each θ-closed set F and each point x ∈ X−F, there exist disjoint U, V ∈ τ such that x ∈ U and F ⊆ V.

Theorem 2.43 For a topological space X, the following properties are equivalent:

1. X is θ-regular;
2. For each θ-open set U in X and each x ∈ U, there exists V ∈ τ such that x ∈ V ⊆ cl(V) ⊆ U;
3. For each θ-open set U in X and each x ∈ U, there exists a clopen set V such that x ∈ V ⊆ U.

Proof. (1) ⇒ (2): Let U be an θ-open set of X containing x. Then X\U is a θ-closed set not containing x. By (1), there exist disjoint X\cl(V), V ∈ τ such that x ∈V and X\U ⊆X\cl(V). Then we have V ∈ τ such that x ∈ V ⊆ cl(V) ⊆ U.

(2) ⇒ (3): Let U be an θ-open set of X containing x. By (2), there exists V ∈ τ such that x ∈V ⊆cl(V) ⊆U. Take V = cl(V). Thus V is closed and so V is clopen. Hence we have V is clopen set such that x ∈V ⊆U.

(3) ⇒ (1): Let F = X\U be a θ-closed set not containing x. Then U is an θ-open set of X containing x. By (3), there exists a clopen set V such that x ∈ V ⊆ U. Then there exist disjoint G = X\V, V ∈ τ such that x ∈ V and F = X\U ⊆ G = X\V. Hence X is θ-regular.

Definition 2.44 A topological space X is said to be θ-β-regular if for each θ-closed set F and each point x ∈ X−F, there exist disjoint U, V ∈ βO(X) such that x ∈ U and F ⊆ V.

Theorem 2.45 For a topological space X, the following properties are equivalent:

1. X is θ-β-regular;
2. For each θ-open set U in X and each x ∈ U, there exists V ∈ βO(X) such that x ∈ V ⊆ βcl(V) ⊆ U;
3. For each θ-open set U in X and each x ∈ U, there exists V ∈ βR(X) such that x ∈ V ⊆ U.

Theorem 2.46 Let f : X → Y be a θ-irresolute β-open almost $\theta_g\beta$-closed surjection. If X is θ-regular, then Y is θ-β-regular.

Proof. Let y ∈Y and V be an θ-open neighborhood of y. Take a point $x \in f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is θ-open in X. By the θ-regularity of X, there exists an θ-open set U of X such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Then y ∈f(U) ⊆f(cl(U)) ⊆ V. Also, since U is open set of X and f is β-open, f(U) ∈ βO(Y). Moreover, since U is β-open, by Theorem 1.3, cl(U) is regular closed set of X. Since f is almost $\theta_g\beta$-closed, f(cl(U)) is $\theta_g\beta$-

closed in Y. Therefore, we obtain $y \in f(U) \subseteq \beta cl(f(U)) \subseteq \beta cl(f(cl(U))) \subseteq V$. It follows from Theorem 2.45 that Y is θ-β-regular.

Corollary 2.47 If $f : X \to Y$ is a θ-irresolute β-open $\theta_g\beta$-closed surjection and X is θ-regular, then Y is θ-β-regular.

Corollary 2.48 If $f : X \to Y$ is a θ-irresolute β-open β-closed surjection and X is θ-regular, then Y is θ-β-regular.

Theorem 2.49 Let $f : X \to Y$ be a θ-irresolute pre-β-open regular $\theta_g\beta$- closed surjection. If X is θ-β-regular, then Y is θ-β-regular.

Proof. Let F be any θ-closed set of Y and $y \in Y - F$. Then $f^{-1}(F)$ is θ-closed in X and $f^{-1}(F) \cap f^{-1}(y) = \theta$. Take a point $x \in f^{-1}(y)$. Since X is θ-β-regular, there exist disjoint sets $U_1, U_2 \in \beta O(X)$ such that $x \in U_1$ and $f^{-1}(F) \subseteq U_2$. Therefore, we have $f^{-1}(F) \subseteq U_2 \subseteq \beta cl(U_2)$, by Theorem 2.34, $\beta cl(U_2) \in \beta R(X)$ and $U_1 \cap \beta cl(U_2) = \theta$. Since f is regular $\theta_g\beta$-closed, by Corollary 2.10, there exists $V \in \beta O(Y)$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq \beta cl(U_2)$. Since f is pre-β-open, we have $f(U_1) \in \beta O(Y)$. Moreover, $U_1 \cap f^{-1}(V) = \theta$ and hence $f(U_1) \cap V = \theta$. Consequently, we obtain $y \in f(U_1) \in \beta O(Y)$, $F \subseteq V \in \beta O(Y)$ and $f(U_1) \cap V = \theta$. This shows that Y is θ-β-regular.

Corollary 2.50 If $f : X \to Y$ is a θ-irresolute pre-β-open pre-$\theta_g\beta$-closed surjection and X is θ-β-regular, then Y is θ-β-regular.

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