STUDY OF THE LOOP GROUPS ACYCLIC MODELS, AND TWISTED TENSOR PRODUCTS

Sudarshan Kumar

Department of Mathematics ,Public Degree College, Bidhuna,Auraiya. U.P

***ABSTRACT** :The main object of this paper is the construction of the Serre spectral sequence by means of twisted tensor products. The basic tool in this development, which is completed in section 32, is Brown's theorem, which gives a natural equivalence between $C_N(F\ X\tau\ B)$ and $C_N(B) \otimes_t C_N(F)$ on the category of twisted Cartesian products, where t is a twisting cochain determined by the twisting function τ. In order to define twisting cochains, the theory of cup, Pontryagin, and cap products is developed in [3]. Of course, the definition of these products depends on the Eilenberg-Zilber theorem, and this is proven in ref [4]. The proofs of bothe Brown's theorem and the Eilenberg-Zilber theorem rely on the method of acyclic models, which is described in ref [5] . The models for Brown's theorem are defined in terms of functions which assign to a (reduced) simplicial set K a simplicial group G(K) and a PTCP G(K) Xτ K such that T(E(τ)) is contractible. G(K) is called a loop group of K. G(K) and G(K) Xτ K are defined in [6] . In ref [7], it is shown that G and $\overline{W}$ are adjoint functions, the suspension E(K) of a complex K is defined, and miscellaneous results about the functions G, $\overline{W}$ and E are obtained.*

INTRODUCTION:

In this paper, we will prove the existence of loop groups of reduced complexes, that is, of complexes having just one vertex. We will also give a reinterpretation of the Hurewicz homomorphism. The method of acyclic models was introduced by Eilenberg and MacLane in [2], and was used in [3] to prove the Eilenberg-Zilber theorem. Our treatment follows these source and MacLane[13]. Most of the material of the last three sections is contained,in rather different form, in Gugenheim [5]. Section 30 relies heavily on suggestions of J.C. Moore. A systematic study of Hopf algebras may be found in Milnor and Moore [9]. The proof of Brown's theorem is parallel to, but simpler than, the topological proof given in his original paper [3]. An explicit expression for a twisting cochain $T(\tau)$ in terms of the twisting function τ has been obtained by Szczarba [12].

The Serre spectral sequence was, of course, studied in the classical paper [10], following its introduction in cohomology by Leray [7, 8]. The approach here shows that the introduction of cubical singular theory is unnecessary, a fact shown by Gugenheim and Moore [6] using quite different methods. Brown [3] proved that the spectral sequence defined here is in fact isomorphic to that defined by Serre. Szczarba [12] studied the products in the Wang spectral sequence using twisted tensor products. The form of d_{n+1} and δ_{n+1} in the case of n-triviality was discovered by Fadell and Hurewicz [4], but of course the result is there proven by Shih in [11].

DEFINITION – 1

A group complex G is said to be loop group of the complex K if there exists a PTCP $E(\tau) = G \times_\tau K$ such that $T(E(\tau))$ is a contractible space.

EXAMPLE:

$K(\pi, n)$ is of course a loop of $K(\pi, n+1)$. By Leema 23.4, if K is a Kan complex, then L(K) is a loop group of K provided that L(K) admits a structure of simplicial group.

If K is a reduced complex and $K_0 = k_0$, we will let K_n denote $s_0^n k_0$. We now define a loop group of such a complex.

DEFINITION – 2 Let K be a reduced complex. Define $G_n(K)$ to be the free group generated by the elements of K_{n+1} modulo the relations $s_0 x = e_n$ for $x \in K_n$. $G_n(K)$ is of course a free group. If $x \in K_{n+1}$, let $\tau(x)$ denote the of x in $G_n(K)$. Define face and degeneracy operators

on G(K) by :

$$\tau(\delta_0 x)\,\delta_0\tau(x) = \tau(\delta_1 x)$$

(T) $$\delta_i\tau(x) = \tau(\delta_{i+1}x) \text{ if } i > 0$$

$$\delta_i\tau(x) = \tau(s_{i+1}x) \text{ if } i \geq 0.$$

The δ_i and s_i extended uniquely to homomorphisms $G_n(K) \longrightarrow G_{n-1}(K)$ and $G_n(K) \longrightarrow G_{n+1}(K)$. G(K) so defined is easily verified to be a group complex, and $\tau : K \rightarrow G(K)$ is clearly a twisting function. We let $E(\tau) = G(K) \times_\tau K$. We must prove that $T(E(\tau))$ is contractible, and it suffices to prove $\pi_1(T(E(\tau))) = 0$ and $\overline{H}_n(E(\tau)) = 0$, $n \geq 0$.

LEMMA1 :

$$\pi_1(T(E(\tau))) = 0.$$

PROOF :Recall that $\pi_1(T(E(\tau)))$ can be considered as a group having one generator for each 1-cell not in a maximal tree and one relation for each 2-cell. We regard non-degenerate simplices $(\acute{g},x)$ as denoting the corresponding cells. The 1-cells $(s_0\acute{g}, x)$, $x \in K_1$ non-degenerate and $\acute{g} \in G_0(K)$, form a maximal tree. This holds since $\delta_0(s_0\acute{g}, x) = (\tau(x)\,\acute{g}, k_0)$ and $\delta_1(s_0\acute{g}, x) = (\acute{g}, k_0)$, which implies that any two 0-cells can be connected in one and only one way by 1-cells of the cited form. We must show that every 1-cell $(\acute{g},x)$, $\acute{g}$ non-degenerate, homotopic to the product of 1-cells in the maximal tree and their inverses(reverses). The 2-cell

($s_1 \acute{g}$, $s_0 x$) shows that ($\acute{g}$,x) is homotopic to ($s_0\delta_0\acute{g}$,x)($\acute{g}$,k_1). If $\acute{g}$ = $\tau(y)^{-1}\acute{g}$, the 2-cell ($s_0\acute{g}$, y) shows that ($\acute{g}$,$\delta_1 y$) is homotopic to ($\acute{g}$,$\delta_0 y$) ($s_0\delta_1\acute{g}$, $\delta_2 y$). If $\acute{g}$ = $\tau(y)\acute{g}$, the 2-cell ($s_0\acute{g}$, y) shows that ($\acute{g}$,$\delta_0 y$) is homotopic to the product of ($\acute{g}$,$\delta_1 y$) $(s_0\delta_1\acute{g}, \delta_2 y)^{-1}$. Combining these relations, ($\acute{g}$,x) is homotopic to the product of ($\acute{g}$,k_1) with 1-cell of the maximal tree or their reverses, where the group theoretic length of $\acute{g}$. Inductively, since

$e_1 = s_0 e_0$, the result is proven.

LEMMA 2 : $\overline{H}_n(E(\tau)) = 0$, $n \geq 0$.

PROOF : Consider $\bar{C}_n(E(\tau))$, where (e_0, k_0) is taken as base point. For $\acute{g} \in G_n(K)$ and $x \in K_{n+1}$, define $[\acute{g},x] \in \bar{C}_n(E(\tau))$ by :

(i) $[\acute{g},x] = (\tau(x)\acute{g}, \delta_0 x) - (\acute{g}, k_n)$.

Observe that $[\acute{g}, k_{n+1}] = 0$ and define $B = \{[\acute{g},x] \,|\, x \neq k_{n+1}\}$. Suppose for the moment that we know that B is a basis for the free Abelian group $\bar{C}_n(E(\tau))$, and define $S : \bar{C}_n(E(\tau)) \to \bar{C}_{n+1}(E(\tau))$ by

(ii) $S[\acute{g},x] = \sum_{i=0}^{n}(-1)^i\,[s_i\acute{g}(s_0)^{i+1}(\delta_1)^i x]$.

Using the easily verified relationsAt this point we have developed all the requisite machinery to define twisting cochains.

DEFINITION 3 : Let $t \in C^1(B;C(G))$, so that $t_q: C_q(B) \to C_{q-1}(G)$.

Define d_t: $C(B) \otimes C(F) \to C(B) \otimes C(F)$ by:

(18) $d_t(b \otimes f) = d(b \otimes f) + t \cap (b \otimes f)$.

Using (8) and (9), we find $d_t^2(b \otimes f) = (\delta(t) + t \cup t) \cap (b \otimes f)$. t is said to be a twisting cochain if $\delta(t) + t \cup t = 0$, that is, if

(19) $dt_n + t_{n-1}d + \sum_{i=1}^{n-1} t_i \cup t_{n-1} = 0$, $n > 1$,

and if $\varepsilon t_1 = 0$ (so that $(\varepsilon \otimes \varepsilon)d_t = (\varepsilon \otimes \varepsilon)(t \cap) = 0$). Then d_t is called the differential twisted by t and C(B) ⊗ C(F) furnished with this differential is denoted by $C(B) \otimes_t C(F)$ and is called a twisted tensor product. Dually, $Hom(C(B) \otimes_t C(F),\Lambda)$ is given the differential δ_t defined by (17) with δ_t and d_t replacing δ and d, or:

(20) $\delta_t(h) = \delta(h) + (-1)^{deg\ h+1} h \cup t$.

· Brown's Theorem

Brown's theorem states essentially that there is a natural way to assign to every twisting function τ a twisting cochain t in such a manner that $C(F\ X_\tau\ B)$ is chain homotopy equivalent to $C(B) \otimes_t C(F)$. In the last section, this result will be used to construct the Serre spectral sequence.

Unless otherwise specified, the symbols C and C^* will denote the normalized chain and cochain functors in this section and the next, and the symbol (n) will refer to formula (n) of section 30.

We will use the method of acyclic models, and we must first define a category, the objects of which are all twisted Cartesian products.

DEFINITION 4 :

Let $F\ X_\tau\ B$ and $F'\ X_{\tau'}\ B'$ be TCP's with groups G and G', and let $\Upsilon: G \to G'$ be a simplicial homomorphism.

A ϒ-map $\theta: E(\tau) \to E(\tau')$ is a simplicial map such that

$$\theta(f,b) = (\psi(b)\alpha(f),\beta(b)),$$

where $\alpha: F \to F'$ is a ϒ-equivalent map, $\beta: B \to B'$ is a simplicial map, and $\psi: B \to G'$ is a function. Clearly $p'\theta = \beta p$. We will write $\theta = (\alpha,\beta,\psi)$. θ is said to be ϒ-

special if $\Upsilon\tau = \tau'\beta$. The requirement that θ be a simplicial map is equivalent to the identities :

$$\tau'\beta(b)\delta_0\psi(b) = \psi(\delta_0 b)y\tau(b)$$

$$(U)\quad \delta_i\psi(b) = \psi(\delta_i b) \quad \text{if } i > 0$$

$$s_i\psi(b) = \psi(s_i b) \quad \text{if } i \geq 0.$$

The composite of a Υ-map θ and Υ'-map θ' is the $\Upsilon'\Upsilon$-map

$\theta'\theta = (\alpha'\alpha,\beta'\beta,(\psi'\beta).(\Upsilon'\psi))$. *With the obvious identity maps, we have defined a category whose objects are all TCP's and whose maps are all Υ-maps. If maps are required to be special, we obtain a subcategory with the same objects, which we shall call $\mathcal{R}$. $\mathcal{P}$ will denote the subcategory of $\mathcal{R}$, the objects of which are all PTCP's. Observe that if $\theta = (\alpha,\beta,\psi)$ is a Υ-map of PTCP's, then necessarily $\alpha = \gamma$. If base complexes are required to be reduced, we obtain subcategory $\mathcal{R}_0$ of $\mathcal{R}$ and $\mathcal{P}_0$ of $\mathcal{P}$. The categories $\mathcal{R}_0$ and $\mathcal{P}_0$ will be of primary interest to us.*

The symbol $F \times_\tau B$ will denote ambiguously an object of

$\mathcal{R}_0$ or the corresponding total complex. We now define model objects in the category $\mathcal{R}_0$. Let $\bar{\Delta}[n]$ denote $\Delta[n]/\Delta[n]^0$, where $\Delta[n]^0$ denotes of $\mathcal{R}_0$ denoted the zero skeleton of $\Delta[q]$, and define the models $M^{p,q}$ of $\mathcal{R}_0$ by :

(i) $M^{p,q} = (G(\bar{\Delta}[p] \times \Delta[q]) \times_{\tau(p)} \bar{\Delta}[p]$.

For clarity, we have here denoted the following twisting function

$\bar{\Delta}[p] \rightarrow G(\bar{\Delta}[p])$

by $\tau(p)$. $G(\bar{\Delta}[p])$ operators on the fibre $G(\bar{\Delta}[p] \times \Delta[q]$ via

$\acute{g}(\acute{g}',u) = (\acute{g}\acute{g}',u)$. *clearly*

(ii) $M^{p,q} = (G(\bar{\Delta}[p] \times_{\tau(p)} \bar{\Delta}[p]) \times \Delta[q])$.

Therefore the realization of each $M^{p,q}$ is contracticible.

Using the models, we can assign a twisting cochain to each twisting function. We first need a definition.

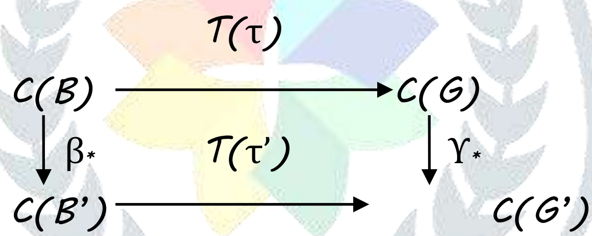
DEFINITION 5 : A twisting cochain on the category $\mathcal{P}_0$ is a function T which assigns a twisting cochain $T(\tau) \in Hom^1(C(B),C(G))$ to each twisting function $\tau : B \rightarrow G$, B a reduced complex, in such a manner that the following conditions are satisfied :

(T.1) $T(\tau)(b) = e_0 - \tau^{-1}(b)$ for all nondegenerate $b \in B_1$

(T.2) If $\tau(b) = e_{q-1}$ for all $b \in B_q$ and all $q \leq n$, then $T(\tau)(b) = 0$

For all nondegenerate $b \in B_q$ and all $q \leq n$.

(T.3) If $\theta = (\gamma,\beta,\psi)$: $G \times_\tau B \rightarrow G' \times_{\tau'} B'$ is a Υ – special map of PTCP's, then the following diagram is commutative :



THEOREM 1 : There exists a twisting cochain on the category $\mathcal{P}_0$.

Proof :- Let $G \times_\tau B$ be PTCP, B areduced complex. Define $T(\tau)_1$ by formula (T.1) and define $T(\tau)_2$ by :

(T.4) $T(\tau)(b) = -\tau^{-1}(b) \cdot s_0\, \tau^{-1}(\delta_0 b)$ for all non-degenerate $b \in B_2$. Clearly $\varepsilon \cdot T(\tau)_1 = 0$, and $dT(\tau)_2 + T(\tau)_1 d + T(\tau)_1 \cup T(\tau)_1 = 0$ is proven by an easy calculation. Condition (T.2) holds for $n=2$ since e_1 is degenerate and therefore zero in $C(G)$. Suppose inductively that $T(\tau)_i$ has been defined for $i < q$, $q > 2$. We require that

$$dT(\tau)_q = -T(\tau)_{q-1}d - \sum_{i=0}^{q-1} T(\tau)_i \cup T(\tau)_{q-i} = X_q \text{ , say,}$$

Clearly $dX_q = 0$ · First consider $G(\bar{\Delta}[q])\ x_{\tau(q)}\ \bar{\Delta}[q]$· since $H_n(G(\bar{\Delta}[q])) = 0$ for $n>0$, there exists $m \in C_{q-1}(G(\bar{\Delta}[q]))$ such that $d(m) = X_q(\Delta_q)$· we define $T(r(q))(\Delta_q) = m$· Next consider $G(K)x_{\tau(k)}K$, $\tau(K) : K \rightarrow G(K)$, K any reduced complex, and let $x \in K_q$ be non-degenerate· $\bar{x}:\bar{\Delta}[q] \rightarrow K$ includes $G(\bar{x}):G(\bar{\Delta})[q])\rightarrow G(K)$ and we define:

$$T(\tau(K))(x) = G(\bar{x})^*(m) \in C_{q-1}(G(K))\cdot \text{ Then we have:}$$

$$dT(\tau(K))(x)= G(\bar{x})^*d(m) = G(\bar{x})^*X_q(\Delta)_q = X_q(\bar{x}(\Delta_q)) = X_q(x)\cdot$$

Finally, consider the arbitrary PTCP $G\ x_\tau\ B$ and let $b \in B_q$ be nondegenerate· If τ is induced by $f(\tau): B \rightarrow \bar{W}(G)$, define :

$$T(\tau)(b) = \Phi(f(\tau))_*[T(\tau(B))(b)] \in C_{q-1}(G), \text{ where } \Phi(f(\tau)): G(B)\rightarrow G \text{ is as}$$

defined in Corollary 27·2· Then again we find $dT(\tau)(b) = X_q(b)$· Condition (T·2) holds since for any $c \in B_q$, $\Phi(f(\tau))(\tau(B)(c)) = \tau(c)$, and since e_{q-1} is degenerate· Clearly condition (T·3) is satisfied, and this completes the proof·

CONCLUSION :We can now define the two functors A and $B_T :\mathcal{R}_0 \rightarrow \mathcal{C}$ that we wish to compare by the method of acyclic models· Thus define $A(F\ x_\tau\ B) = C(F\ x_\tau\ B)$ and $B_T(F\ x_\tau\ B) = C(B) \otimes_t C(F)$, where $t = T(\tau)$, T being a fixed twisting cochain on the category $\mathcal{P}_0$· Observe that $\theta = (\alpha,\beta,\psi)$ is a Υ-special map of TCP's, then (T·3) and (10) of above Lemma guarantee that $B_T(\theta) = \beta_* \otimes \alpha_*$ is a chain map·

BILIOGRAPHY

[1] "Twisted tensor products I," Ann· Of Math·, 69(1959), 223-246·

[2] S· EILENBERG and S· MACLANE· "Acyclic Models," Am· J· Math·, 7591953), 189-199·

[3] "On products of complexes," Am· J· Math·, 75(1953), 200-204·

[4] E.FADELL and W. HUREWICZ. "On the structure of higher differential operators in spectral sequence," Ann. Of Math. (2), 68(1985), 314-347.

[5] "On a theorem of E.H. Brown," Ill. J. Math., 4(1960), 292-311.

[6] V.K.A.M. GUGENHEIM and J.C. MOORE. "Acyclic models and fibre spaces," Trans. A.M.S., 85(1957), 265-306.

[7] J. LEARY. "L'anneau spectral et l'anneau d'homologie d'un espace localement compact et d'une application continue," J. Math. Pures Appl., 29(1950), 1-139.

[8] "L'homologie d'un espace fibre don't la fibre est connexe," J. math. Pures Appl., 29(1950), 169-213.

[9] J. MILNOR and J.C. MOORE. "On the structure of Hopf algebras," Ann. of Math., 81(1965), 211-264.

[10] J.P. SERRE. "Homologie singuliere des espaces fibres," Ann. of Math., 54(1951), 425-505.

[11] W. SHIH. Homologie des Espaces Fibres, Institute des Hautes Etudes Scientifiques, Publication mathematiques no. 13(1962).

[12] R.H. SZCZARBA. "The homology of twisted Cartesian products," Trans. A.M.S., 100(1961), 197-216.

[13] Homology, New York, Academic Press, 1963.