ACOUSTICS BEYOND LINEARITY

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Abstract: There is a classical effect on acoustic waveform when the sound propagates through a nonlinear medium. The motivation of this work was to experimentally verify and further study the phenomena of nonlinear waveform distortion. Measurements were conducted in water with a 0.5" -diameter unfocused transducer, working at 3.5-MHz peak frequency as a source. We use a membrane hydrophone with an active sensor diameter of 0.2mm with a frequency response 1-30MHz as a detector. Amplitudes of fundamentals and harmonics are measured, and hence nonlinearity of the medium is determined. The experimental results agreed with the modeling based on the Khokhlov Zabolotskaya Kuznetskov (K.Z.K.) equation.

***Index Terms* - Ultrasonics, Nonlinear medium, Waveform distortion, K.Z.K. Equation, Experimental harmonic detection.**

I. INTRODUCTION

Linear acoustics has explained in detail different phenomena like propagation, reflection, transmission, refraction, diffraction, absorption, and dispersion. But our understanding of nonlinear acoustics is quite limited; although nonlinear acoustics has developed in the past 40 years, the field itself is old, and its development has continued for the last 200 years. There are two main reasons. First, ordinary linear acoustics does an outstanding job of explaining most of the acoustical phenomena; that's why the necessity of nonlinear acoustical theory did not arise. The second cause is nonlinear mathematics necessary to describe the finite-amplitude sound is very difficult to handle. Because of the efforts of Burgers, Hopf, Cole, and Lighthill, the once-challenging problem of finite amplitude propagation in dissipative fluids is no longer a mystery. And thanks to Westervelt, Khokhlov, Zabolotskaya, Kuzenetskov, Beyer, Blackstock, and Hamilton — who have opened the door.

II. THEORY

2.1 Origin of Nonlinearity

$$\frac{dx}{dt} = c_0 \pm \beta u$$

Here, $\frac{dx}{dt}$, denotes the speed of propagation of the wave. c_0 is the small signal sound speed, u is the velocity of fluid particles, and β is the nonlinearity coefficient. As a result, the wave gets distorted during propagation. The factor β determines this distortion, called the nonlinear parameter (Blackstock, 2000, Hamilton, 1998).

2.2 Nonlinear parameter $\beta = (1 + B/2A)$

We express the pressure amplitude p in terms of ϱ as

$$p = A\left(\frac{\varrho'}{\varrho_0}\right) + \frac{B}{2!}\left(\frac{\varrho'}{\varrho_0}\right)^2 + \frac{C}{3!}\left(\frac{\varrho'}{\varrho_0}\right)^3 + \cdots$$

where, $A = \varrho_0\left(\frac{\partial p}{\partial \varrho}\right)_{s,0} \equiv \varrho_0 c^2$ and $B = \varrho_0^2\left(\frac{\partial^2 p}{\partial \varrho^2}\right)_{s,0}$.

2.3 Conservation Laws

$$\frac{\partial \varrho}{\partial t} + \vec{\nabla}\cdot\{\varrho \ \vec{u}\} = 0 \quad \text{(Mass Conservation)}$$

$$\frac{\partial(\varrho \ \vec{v})}{\partial t} + \vec{\nabla}\cdot(\varrho \ \vec{v}) \ \vec{v} + \vec{\nabla}p = 0 \quad \text{(Momentum Conservation)}$$

III. MODEL EQUATION

If we consider viscous effects due to the medium of sound propagation, the momentum equation will look like this —

$$\frac{\partial \rho \vec{v}}{\partial t} + \vec{\nabla}\left(\overleftrightarrow{P} + \overleftrightarrow{PI} + \overleftrightarrow{D}\right) = 0 \qquad (1)$$

Where

$$\overleftrightarrow{P} = \textbf{momentum tensor} = \begin{bmatrix} \varrho v_x v_x & \varrho v_x v_y & \varrho v_x v_z \\ \varrho v_y v_x & \varrho v_y v_y & \varrho v_y vz \\ \varrho v_z v_x & \varrho v_z v_y & \varrho v_z v_z \end{bmatrix} \qquad (2)$$

$\overleftrightarrow{PI}$ =pressure tensor where $\overleftrightarrow{I}_{ij} = \delta_{ij}$, and $\overleftrightarrow{D}$ = viscous-stress tensor where

$$\overleftrightarrow{D}_{ii} = -\left(\eta - \frac{2}{3}\mu\right)\vec{\nabla}\cdot\vec{v} - 2\mu\frac{\partial v_i}{\partial x_i} \qquad (3) \text{ and}$$

$$\overleftrightarrow{D}_{ij} = -\mu\frac{\partial v_i}{\partial x_j} - \mu\frac{\partial v_j}{\partial x_i} \qquad i \neq j \qquad (4)$$

Now suppose any field variables ψ, ϱ and $\vec{v}$ can be expressed by the unperturbed values, ϱ_0 and $\vec{v}_0$ with their higher order perturbations as

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots \qquad (5)$$

$$\varrho = \varrho(\overleftrightarrow{P}) = \varrho_0 + \varrho_1 + \varrho_2 + \cdots \qquad (6)$$

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots \qquad (7)$$

Here we assumed the velocity of the unperturbed fluid is zero ie $\vec{v}_0 = 0$. All those substitutions in equations(1) & (3) will give the first-order correspondence-

$$\frac{\partial \varrho_1}{\partial t} + \varrho_0 \overrightarrow{\nabla}\cdot\vec{v}_1 = 0 \qquad (8)$$

$$\varrho_0\frac{\partial \vec{v}_1}{\partial t} + \vec{\nabla}\left(\overleftrightarrow{P}_1 I + \overleftrightarrow{D}\right) = 0 \qquad (9)$$

$$\varrho_1 = \left(\frac{\partial \varrho}{\partial P}\right)_0 \overleftrightarrow{P}_1 = \frac{1}{c^2}\overleftrightarrow{P}_1 \qquad (10)$$

If we use the expression of $\overleftrightarrow{D}$ from equation (4), we can eliminate vi from these equations, and finally, we can get after time derivative of equation(8) and divergence of equation(9), respectively, for the 1st-order pressure field result —

$$\vec{\nabla}^2\left(\overleftrightarrow{P}_1 + \frac{v'}{c^2}\frac{\partial \overleftrightarrow{P}_1}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2} = 0 \qquad (11)$$

Where, $\varrho v' = \frac{4\mu}{3} + \eta$, μ, and η being the shear and bulk viscosities, respectively. If we don't consider any viscous contribution in this equation, the equation will give the form of a plain wave equation of 1st order. and i.e.,

$$\vec{\nabla}^2\left(\overleftrightarrow{P}_1\right) - \frac{1}{c^2}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2} = 0 \qquad (12)$$

For the 2nd-order case, we obtain the following equations

$$\frac{\partial \varrho_2}{\partial t} + \vec{\nabla}\cdot\varrho_0\vec{v}_2 = -\vec{\nabla}\cdot\varrho_0\vec{v}_1 \qquad (13)$$

$$\varrho_0\frac{\partial \vec{v}_2}{\partial t} + \vec{\nabla}\cdot\overleftrightarrow{P}_2 + \vec{\nabla}\cdot\overleftrightarrow{D}_2 = -\vec{\nabla}\cdot\overleftrightarrow{P}_2 - \frac{\partial \varrho_1\vec{v}_1}{\partial t} \qquad (14)$$

$$\varrho_2 = \left(\frac{\partial \varrho}{\partial P}\right)_0 \overleftrightarrow{P}_2 + \frac{1}{2}\left(\frac{\partial^2 \varrho}{\partial P^2}\right)_0 \overleftrightarrow{P}_1^2 = \frac{1}{c^2}\overleftrightarrow{P}_2 + \frac{\Gamma}{\varrho_0 c^2}\overleftrightarrow{P}_1^2 \qquad (15)$$

Here $\Gamma = \frac{1}{2}\varrho_0 c_0^4\left(\frac{\partial^2 \varrho}{\partial P^2}\right)_0$ For ideal gas (where $P\varrho^{-\gamma} = constant$), $\Gamma = -\frac{(\gamma - 1)}{2\gamma}$.

The corresponding wave equation for $\overleftrightarrow{P}_2$ is

$$\vec{\nabla}^2\left(\overleftrightarrow{P}_1 + \frac{v'}{c^2}\frac{\partial \overleftrightarrow{P}_1}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2} = -\vec{\nabla}\cdot\overleftrightarrow{P}_2\cdot\vec{\nabla} \qquad (16)$$

$$-\frac{\Gamma}{\varrho_0 c_0^4}\frac{\partial}{\partial t}\left(\frac{\partial \overleftrightarrow{P}_1^2}{\partial t} + v'\vec{\nabla}^2\overleftrightarrow{P}_1^2\right) \qquad (17)$$

Now, if we set the following transformations

$$\overleftrightarrow{P}_2 = \overleftrightarrow{P}_2' - \frac{\varrho_0 v_1^2}{2} - \frac{\overleftrightarrow{P}_1^2}{2\varrho_0 c_0^2} - \frac{1}{\varrho_0 c^2}\frac{\partial \overleftrightarrow{P}_1}{\partial t}\int \overleftrightarrow{P}_1 dt \quad (18)$$

$$\vec{v}_2 = \vec{v}_2' - \frac{1}{\varrho_0 c_0^2}\frac{\partial}{\partial t}\left(\vec{v}_1 \int \overleftrightarrow{P}_1 dt\right) \quad (19)$$

$$\varrho_2 = \varrho_2' - \frac{\varrho_0 v_1^2}{2c_0^2} - (1-\Gamma)\frac{\overleftrightarrow{P}_1^2}{2\varrho_0 c_0^4} - \frac{1}{\varrho_0 c_0^4}\frac{\partial \overleftrightarrow{P}_1}{\partial t}\int \overleftrightarrow{P}_1 dt \quad (20)$$

Equation (16) will change into:

$$\vec{\nabla}^2 \overleftrightarrow{P}_2' - \frac{1}{c^2}\frac{\partial^2 \overleftrightarrow{P}_2'}{\partial t^2} = -\frac{(1-\Gamma)}{\varrho_0 c_0^4}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2} \quad (21)$$

IV. Quasilinear Solution

The governing equation

$$\vec{\nabla}^2 \overleftrightarrow{P}_2' - \frac{1}{c^2}\frac{\partial^2 \overleftrightarrow{P}_2'}{\partial t^2} = -\frac{(1-\Gamma)}{\varrho_0 c_0^4}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2} \quad (22)$$

It's evident in the above that the 1st order wave solution $\overleftrightarrow{P}_1(\vec{r},t)$of the nonlinear medium appears as a contributing source of secondary wave $\overleftrightarrow{P}_2$ and so on for other solutions.

Equation(6) can be rewritten as

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\overleftrightarrow{P}_2' = -4\pi Q \quad (23)$$

Where $Q = \frac{(1-\Gamma)}{4\pi\varrho_0 c_0^4}\frac{\partial^2 \overleftrightarrow{P}_1}{\partial t^2}$is time retarded greens function. And the time retarded solution will be

$$\overleftrightarrow{P}_2' = \int_V \frac{Q\left(t - \frac{|\vec{r}|}{c_0}\right)dV}{r} \quad (24)$$

$$\overleftrightarrow{P}_1(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (25)$$

Therefore

$$Q\left(\vec{r},\tau = t - \frac{|\vec{r}|}{c_0}\right) = \frac{A_0^2\omega^2}{\pi}e^{2i\left(\vec{k}\cdot\vec{r}-\omega t+\frac{\omega|\vec{r}|}{c_0}\right)} \quad (26)$$

And finally

$$\overleftrightarrow{P}_2 = Ae^{i(\vec{k}'\cdot\vec{r})} + \frac{4A^2\omega^2}{k'^2-4k^2}e^{i(2\vec{k}\cdot\vec{r}-2\omega t)} \quad (27)$$

Where $k' = \frac{2\overleftrightarrow{P}(0)\omega}{c_0}$.

V. Parametric Array

We can consider first-order pressure P_1 consists of two harmonic components, P_a and $P_b$with frequencies ω_a and ω_b. Pressure P_1 then

$$P_1 = Re(P_a + P_b) = \frac{1}{2}\left[\left(P_a + P_a^*\right) + \left(P_b + P_b^*\right)\right] \quad (28)$$

Where $P_a{}^*$ is the complex conjugate of P_a. Then if we express the solution $P'_{2\pm}$ of the equation - that corresponds to the frequency $\omega_\pm$ as $Re(P_\pm)$, the equation for $P_\pm$ is

$$\left(\vec{\nabla}^2 + k_\pm^2\right)P_\pm = \frac{1-\Gamma}{\varrho_0 c^2}k_\pm^2 P_a P_{b\pm} \quad (29)$$

where $k_+ = \omega_+/c = (\omega_a + \omega_b)/c$ and $P_{b+} = P_b P_{b-} = P_b^*$.

For two plane waves traveling in the same direction along the x axis, we have $P_a = Ae^{(ik_a x - i\omega_a t)}$ and $P_b = Ae^{(ik_b x - i\omega_b t)}$ where A and B are real. Then the equation for $P_\pm$ is

$$\left(\vec{\nabla}^2 + k_\pm^2\right) = \frac{1-\Gamma}{\varrho_0 c^2} k_\pm^2 ABe^{(ik_\pm x - i\omega_\pm t)} \quad (30)$$

$$\equiv C_\pm e^{(ik_\pm x - i\omega_\pm t)} \quad (31)$$

The particular solution, for the plus sign, is

$$P'_{2+} = \mathrm{Re}(P_+) = \mathrm{Re}\left(\frac{C_+ x}{2ik_+}\right)e^{(ik_+ x - i\omega_+ t)} \quad (32)$$

$$= \frac{1-\Gamma}{2\varrho_0 c^2} AB(k_+ x) sin(k_+ x - \omega_+ t) \quad (33)$$

$$= \frac{(1-\Gamma)\varrho_0}{2} v_a v_b (k_+ x) sin(k_+ x - \omega_+ t) \quad (34)$$

where v_a and v_b are the velocity amplitudes of the two primary waves.

VI. KZK Equation

The nonlinear acoustical field radiated by a circular piston source can be modeled (Lee, 1995) by the simplest nonlinear Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation, including combined effects of attenuation and diffraction. KZK equation can be expressed in terms of the axial component as:

$$\frac{\partial}{\partial r}\left[\frac{\partial p}{\partial z} - \frac{\beta}{2\varrho_0 c^2}\frac{\partial p^2}{\partial r} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial r^2}\right] = \frac{c}{2}\Delta_\perp p$$

Here, $\vec{u}$ is the particle velocity, $\tau = t - z/c$ is the retarded time, ϱ_0 is the ambient density, β is the nonlinear parameter, and b is the dissipative parameter of the medium. $\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ is the transverse Laplacian, and r is the lateral coordinate (distance from the axis of symmetry). Weak finite amplitude radiation from axisymmetric sources will cause second harmonic, sum-frequency and different frequency generation. The method of successive approximations will give us the expressions. Firstly, we are just considering the second harmonic generation by an acoustic beam radiated at a single angular frequency ω. We can assume a quasilinear solution of the form

$$p = p_1 + p_2 \quad (35)$$

where p_1 is the linear solution and p_2 second order correction term over p_1 $(|p_2| << |p_1|)$. The space-dependent term is now separated from the time dependence:

$$p_n(r,z,\tau) = \frac{1}{2j} q_n(r,z) e^{jn\omega\tau} + c.c. \qquad n = 1,2 \quad (36)$$

where q_n are complex amplitudes, and cc denotes the complex conjugate terms. Assuming the second harmonic component is absent in the source radiation at the transducer surface $(z=0)$,therefore $q_2(r,0) = 0$ .Now the KZK equation will give two equation

$$\frac{\partial q_1}{\partial z} + \frac{j}{2k}\nabla_\perp^2 q_1 + \alpha_1 q_1 = 0 \quad (37)$$

$$\frac{\partial q_2}{\partial z} + \frac{j}{2k}\nabla_\perp^2 q_2 + \alpha_2 q_2 = \frac{\beta k}{2\varrho_0 c_0^2} q_1^2 \quad (38)$$

where $\alpha_n = \delta n^2 \omega^2 / 2c_0^3$, the attenuation coefficient at frequency $n\omega$, and k the wave number. With the proper Green's function, the solution will be:

$$q_1(r,z) = 2\pi \int_0^\infty q_1(r',0) G_1(r,z|r',z') r' dr' \quad (39)$$

$$q_2(r,z) = \frac{\beta k \pi}{\varrho_0 c_0^2} \int_0^z \int_0^\infty q_1^2(r',z') G_2(r,z|r',z') r' dr' dz' \quad (40)$$

while

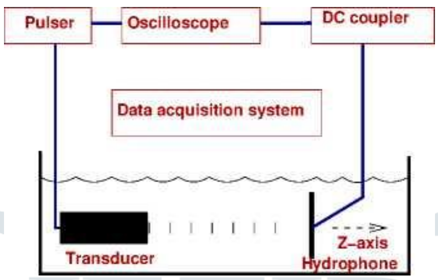
$$G_n(r,z|r',z') = \frac{jnk}{2\pi(z-z')} J_0\left(\frac{nkrr'}{z-z'}\right) e^{\left[-\alpha_n(z-z') - \frac{jnk(r^2+r'^2)}{2(z-z')}\right]} \quad (41)$$

Injection of known source function $q_1(r',0)$, we can obtain the second harmonic component. For Gaussian source with characteristic radius $a$with an ambient pressure P_0, the source function can be expressed as:

$$q_1(r',0) = P_0 e^{-\left(\frac{r}{a}\right)^2} \quad (42)$$

VII. Experiment

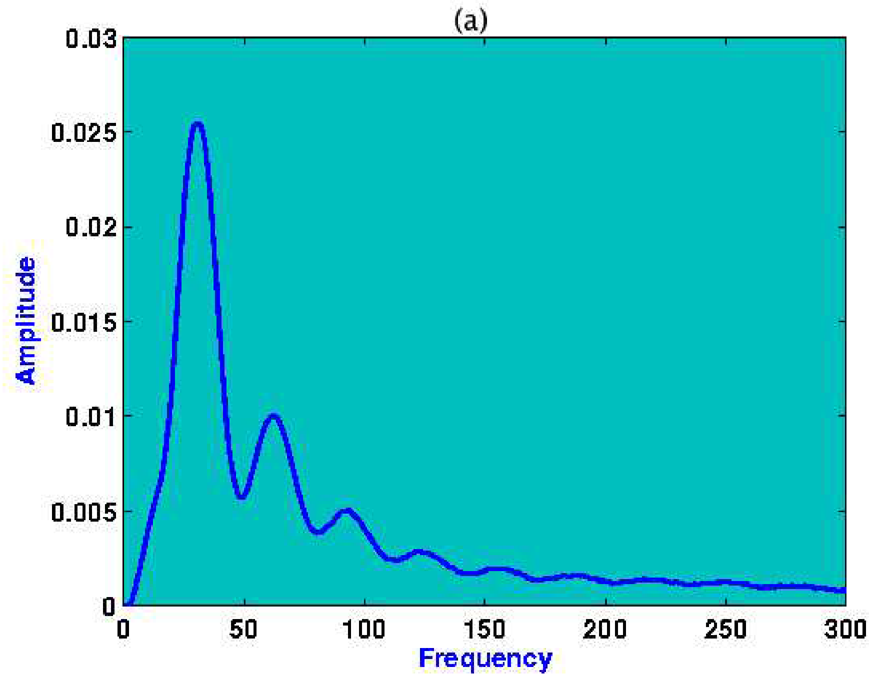
We begin with a piston-type transducer as an acoustic source of a fundamental frequency of 3.5MHz and an effective diameter of 0.5". We have used an immersion-type piston transducer from Panametrics, USA, model number V382. A membrane hydrophone of an active sensor diameter of 0.2mm with a frequency response 1-30MHz (Precision acoustics, UK) has been used as a detector. The transducer was placed in a water tank of dimension filled with millipore water (Labat, 2000, Oleg, 2004). The membrane hydrophone was placed in the moving arm of an x-y-z computer-controlled position system.



Water Tank
Figure 1: Schematic diagram of the experimental arrangements

VIII. Experimental Results and Discussion

Details study of the nonlinear KZK equation in the quasilinear approach has allowed us to be motivated for further resear and to do some relevant experiments. And we did so. As the system is nonlinear, a numerical study is also opened. We have traveled these three fields simultaneously and quite successfully. The nonlinear behavior of ultrasound radiated by a piston source in a medium is studied theoretically, experimentally, and numerically. The source differs from a plane wave. In the distance-amplitude profile for different frequencies, it's pretty transparent that nonlinearity is a cumulative property of the medium. Because the initial amplitudes for harmonics are very small, but for fundamental, it's a trad value, far from zero. Experimentally we have calculated nonlinear parameters for millipore water (B/A=4.0098, Dissolved oxygen content= 9%) at room temperature. This measured value of the B/A parameter is also not beyond the common value. We have cal¬culated the axial pulse intensity integral (PII) for different frequencies. Now we have the freedom and opportunity to use this PII to contribute to the field of nonlinear acoustics. Spatial symmetry breaking has been studied in the transverse plane. If we look into the contour profile for different dB scales, symmetry has been broken. This phenomenon must be studied and verified from a theoretical point of view.



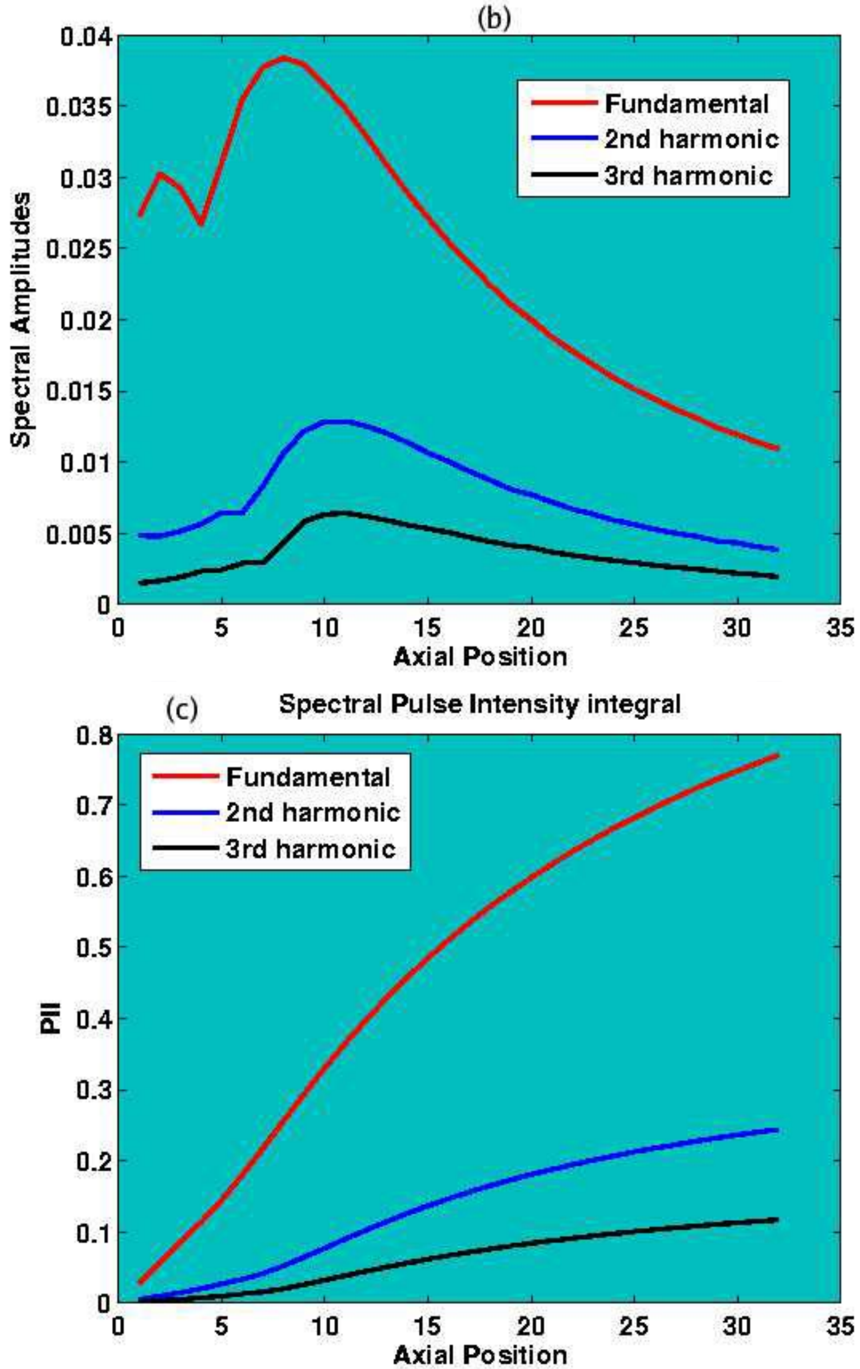


Figure 2: (a)Power spectrum at its 18th position. (b) Amplitude profile for different harmonics. (c) PII in axial direction for different harmonics

The scope of future research are to study of nonlinear behavior of ultrasound in different bio-mimicking mediums, especially tissues, theoretical study of spatial and temporal symmetry breaking, numerical code development of model equation, tissue harmonic imaging, and theoretical and experimental study of shock wave propagation and application in medium.

IX. Acknowledgment

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