CUSTOMER INDUCED INTERRUPTION AND SELF-GENERATION OF PRIORITIES IN A SINGLE SERVER QUEUEING MODEL

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Abstract

In this article we consider an infinite capacity single server queueing model to which customers' arrival is according to Poisson Process. At the time of arrival, all customers are classified as ordinary. If the server is busy, the arriving customers join a queue. Waiting customers generate priority at constant rate γ. Such customers wait in a reserved waiting space of capacity 1. We consider a customer induced interruption while service is going on. The interruption occurs according to a Poisson process. The interrupted customers will enter into a buffer B_1 of finite capacity K. The interrupted customers will spend a random period of time for completion of interruption. The duration of interruption with s customers in B_1 follows an exponential distribution. The service facility consists of one server and duration of service times of ordinary, priority and interruption completed customers follows exponential distribution with different parameters. Various performance measures are obtained and suitable cost function for getting optimal buffer size K is also derived.

Keywords:

Self-generation of priorities, Customer induced interruption, Level Dependant Quasi-Birth-Death Process, Matrix Analytic Method.

Introduction

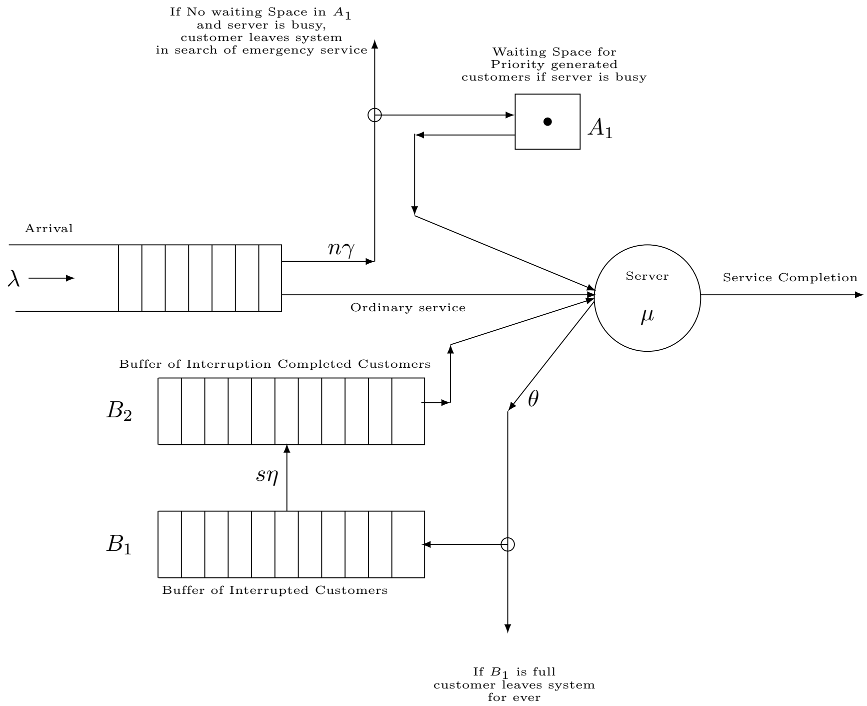
Queueing models with interruption have a significant role in the modern digital world of communication networks like Internet, cell phones etc. Service interruption models studied in the literature include server break-downs, arrival of priority customer, customer induced interruption etc. Customer induced interruptions are not so common compared to other types of server interruptions. But it is possible in doctors clinic; especially in organ transplanting patients, health maintenance offices and on line services. Two remarkable features of customer induced interruption compared with service interruption are:

(i) there can be more interrupted customers than number of servers in the system,

(ii) the system can offer services to other customers while customer induced interruption takes place. For example,in a doctors clinic, while a patient is being examined, the physician may find that one or more tests are needed for prescribing exact medicine. Hence patient is asked to under go these tests and return to clinic. Such patient scan be considered as self interrupted customer. Another example is the case of the customer getting interrupted by an emergency call during his cellphone conversation which needs immediate attention.

Model Description

We consider an infinite capacity single server queueing model to which customers' arrival is according to Poisson process with rate λ.



At the time of arrival, all customers are classified as ordinary. If the server is busy, the arriving customers join a queue. Waiting customers *generate priority* at constant rate γ in such a way that if there are n customers in the queue then the rate of priority generation is $n\gamma$. Such a customer waits in waiting space A_1 of capacity 1 (exclusively for priority generated customers) for service which begins on completion of the present service. A second customer who generates priority during that time period (while the previously generated priority customer is waiting) will leave the system in search of emergency service. We consider a customer induced interruption while service is going on. The interruption occurs according to a Poisson process of rate θ. When interruption occurs the customer currently in service will force to leave the service facility. The freed server is ready to offer services to other customers. The interrupted customers will enter into a buffer B_1 of finite capacity K if space is available. Otherwise, such a customer is lost forever. The interrupted customers will spend a random period of time for completion of interruption. The duration of interruption with s customers in B_1 follows an exponential distribution with rate $s\eta$. The service facility consists of one server and service times are exponentially distributed with parameter μ_0, μ_1, μ_2 for ordinary, priority generated and B_2 customers respectively. Here we assume that priority generated customers never undergo interruption and not more than one interruption is allowed for a customer while in service. That is, an interrupted customer who gets into service again will leave the system with no further interruption. The interrupted customers upon completing their interruptions enter into a buffer B_2 whose size is K. Customers in B_2 buffer are given non-preemptive priority over new customers. Thus a free server will offer services either to those customers waiting in B_2 or new customers. Because of the restriction together with the fact that atmost one interruption is allowed for any customer, the total number of customers in B_2 will never exceed the size of B_1 and hence we assume buffer sizes to be the same. Also assume that the sum of buffer sizes should be less than or equal to K. Otherwise, if buffer B_2 is full, we cannot accommodate one more interruption-completed customer from B_1.

The model is studied as a quasi birth-death (QBD) process and matrix geometric type solution is obtained. To write the state space of QBD we use the following notations: Let

$N_1(t)$=Number of ordinary customers in the queue(including the one,if any, in service) at time t.

$N_2(t)$=Number of customers in buffer B_2 at time t.

$N_3(t)$=Number of interrupted customers in buffer B_1 at time t.

$$N_4(t)= \begin{cases} 0, & \text{when server is busy with ordinary customer} \\ 1, & \text{when server is busy with priority customer} \\ 2, & \text{when server is busy with customer in } B_2 \end{cases}$$

$N_5(t)$=Number of priority generated customers waiting for service at time t.

Under the assumptions on arrival and service processes $X(t) = \{(N_1(t), N_2(t), N_3(t), N_4(t), N_5(t)) : t \geq 0\}$ form a continuous time Markov chain on the state space $\Omega = \{(n, i, j, m_1, m_2) : n \geq 0; i, j = 0, 1, 2, \cdots, K;\ i + j \leq K;\ m_1 = 0, 1, 2;\ m_2 = 0, 1\}$ Here the levels are according to the number of ordinary customers in the system. The generator of the above Markov chain is of the form

$$Q = \begin{bmatrix} B_0 & A_1 & & & & & \cdots \\ C_1 & B_1 & A_1 & & & & \cdots \\ & C_2 & B_2 & A_1 & & & \cdots \\ & & \ddots & \ddots & \ddots & & \\ & & & C_m & B_m & A_1 & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_1, B_0, B_i, C_i,\ i = 1, 2, 3, \cdots$; are of order δ , where $\delta = 3(K + 1)(K + 2)$ and defined as follows:

☐ Matrix B_0 has the following transitions;

- $(0, b_2, b_1, 1, 1) \xrightarrow{\mu_1} (0, b_2, b_1, 1, 0)$; where $b_2, b_1 = 0, 1, 2, 3, \ldots K; b_2 + b_1 \leq K$
- $(0, b_2, b_1, 2, 0) \xrightarrow{\mu_2} (0, b_2 - 1, b_1, 2, 0)$; where $b_2 = 1, 2, 3, \ldots K; b_1 = 0, 1, 2, 3, \ldots K; b_2 + b_1 \leq K$
- $(0, x, y, s, w) \xrightarrow{y\eta} (0, x+1, y-1, s, w)$; where $w = 0, 1; s = 0, 1, 2; y = 1, 2, \ldots K; x = 0, 1, 2 \ldots K; x + y \leq K$
- We note that the entries in any row of this matrix add to zero. Hence diagonal entry is equal to the negative of the sum of the other entries in that row.

  ☐ Matrix A_1 has the following transitions;

- $(i, b_2, b_1, s, w) \xrightarrow{\lambda} (i+1, b_2, b_1, s, w)$; where $i \geq 1; w = 0, 1; s = 0, 1, 2\ ; b_2, b_1 = 0, 1, 2, 3, \ldots K;\ b_2 + b_1 \leq K$

  ☐ Matrix C_i has the following transitions;

- $(i, b_2, b_1, 0, 0) \xrightarrow{\mu_0} (i - 1, b_2, b_1, 0, 0)$; where $i \geq 1; b_2, b_1 = 0, 1, 2, 3, \ldots K - 1; b_2 + b_1 \leq K$
- $(i, 0, K, 0, 0) \xrightarrow{\mu_0 + \theta} (i - 1, 0, K, 0, 0)$; where $i \geq 1; b_2 = 1, 2, 3, \ldots K; b_1 = 0, 1, 2, 3, \ldots K; b_2 + b_1 \leq K$
- $(i, b_2, b_1, s, w) \xrightarrow{\theta} (i - 1, b_2, b_1 + 1, s, w)$; where $i \geq 1; w = 0, 1; b_2 = 0, 1, \ldots, K; b_1 = 0, 1, \ldots, K - 1; b_2 + b_1 \leq K$.
- $(i, 0, K, 0, 1) \xrightarrow{i\gamma + \theta} (i - 1, 0, K, 0, 1)$; where $i \geq 1$.

- $(i, b_2, b_1, s, 0) \xrightarrow{i\gamma} (i-1, b_2, b_1, s, 1)$; where $i \geq 1; s = 0, 1, 2; b_1, b_2 = 0, 1, \ldots, K; b_2 + b_1 \leq K$
- $(i, b_2, b_1, s, 1) \xrightarrow{i\gamma} (i-1, b_2, b_1, s, 1)$; where $i \geq 1; s = 0, 1, 2; b_1, b_2 = 0, 1, \ldots, K; b_2 + b_1 \leq K$

  □ Matrix B_i has the following transitions;

- $(i, b_2, b_1, 1, 1) \xrightarrow{\mu_1} (i, b_2, b_1, 1, 0)$; where $i \geq 1; b_2, b_1 = 0, 1, 2, 3, \ldots K; b_2 + b_1 \leq K$
- $(i, b_2, b_1, 2, 0) \xrightarrow{\mu_2} (i, b_2 - 1, b_1, 2, 0)$; where $i \geq 1; b_2 = 1, 2, 3, \ldots K; b_1 = 0, 1, 2, 3, \ldots K; b_2 + b_1 \leq K$
- $(i, x, y, s, w) \xrightarrow{y\eta} (i, x+1, y-1, s, w)$; where $i \geq 1; w = 0, 1; s = 0, 1, 2; y = 1, 2, \ldots K; x = 0, 1, 2 \ldots K; x + y \leq K$
- We note that the entries in any row of this matrix add to zero. Hence diagonal entry is equal to the negative of the sum of the other entries in that row.

System Stability

The distinctive nature of self generation of priorities of above process gives the intuition that the process $X(t)$ will be ergodic. Since the priority generation is at a linear rate and departure of such customers is without taking service, the system turns out to be stable as is proved in the following theorem.
Theorem *:.* The system under discussion is always stable.

Proof. Drift is negative except for finite number of stages. Consider the *Lyapunov* test function defined by $\phi(s) = i$ where s is a state in level i. For a state s in level i , the mean drift y_s is given by

$$\begin{aligned} y_s &= \sum_{p \neq s} [\phi(p) - \phi(s)] q_{sp} \\ &= \sum_{s'} [\phi(s') - \phi(s)] q_{ss'} + \sum_{s''} [\phi(s'') - \phi(s)] q_{ss''} + \sum_{s'''} [\phi(s''') - \phi(s)] q_{ss'''} \end{aligned}$$

where s', s'', s''' vary over states belonging to levels $i-1, i$ and $i+1$ respectively. Then $\phi(s) = i,\ \phi(s') = i - 1,\ \phi(s'') = i,\ \phi(s''') = i + 1$

$$y_s = \sum_{s'} q_{ss'} + \sum_{s'''} q_{ss'''}$$

Since $\sum_{s'''e} q_{ss'''}$ is bounded by some fixed constant for any s in level $i \geq 1$ we can find a positive real number K such that $\sum_{s'''} q_{ss'''} < K$ for all s in level $k \geq 1$. Thus for any $\epsilon > 0$, we can find K^* large enough that $y_s < -\epsilon$ for any s belonging to level $i \geq K^*$. Hence the theorem follows from Tweedie's [6] result. □

Steady State Distribution

Let $x = (x_0, x_1, x_2, \cdots)$ be the equilibrium distribution. For a positive recurrent level dependent quasi-birth-death (LDQBD) process, x_i satisfies the relationship

$$x_{k+1} = x_k R_k \ , \quad k \geq 0$$

which gives

$$x_{k+1} = x_0 \prod_{l=0}^{l=k} R_l$$

where the family of matrices $\{R_k : \; k \geq 0\}$ are the minimal non-negative solution of the system of equations:

$$A_1 + R_0 B_1 + R_0 R_1 C_2 = 0 \tag{1}$$

$$A_1 + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \tag{2}$$

for $k \geq 1$ and x_0 is the solution of

$$x_0(B_0 + R_0 C_1) = 0 \tag{3}$$

subject to

$$x_0 e + x_0 (\sum_{j=1}^{\infty} \prod_{l=0}^{j-1} R_l) e = 1 \tag{4}$$

Before we pass on to the numerical computations we construct a dominating process. $\{X(t) \;,\; t \geq 0\}$ satisfies the condition that for all $k \geq 1$ and for all i, there exists j such that $(C_k)_{ij} > 0$.

Therefore there exists a dominating process $\bar{X(t)}$ on the same statespace of $X(t)$ with generator

$$\bar{Q} = \begin{bmatrix} B_0 & A_1 & & & & & \cdots \\ 0 & \bar{B_1} & \bar{A_1} & & & & \cdots \\ & \bar{C_2} & \bar{B_2} & \bar{A_1} & & & \cdots \\ & & \ddots & \ddots & \ddots & & \\ & & & \bar{C_k} & \bar{B_k} & \bar{A_1} & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$\begin{aligned} (\bar{A_1})_{ij} &= \frac{1}{\delta}((A_1 e)_{max}) \\ (\bar{C_k})_{ij} &= \frac{1}{\delta}((C_{k-1} e)_{min}), \; k \geq 2 \\ (\bar{B_k})_{ij} &= (B_k)_{ij}, \; i \neq j \;, \quad k \geq 1 \end{aligned}$$

with $((A_1 e)_{max})$ is the maximum element of the coloum vector $A_1 e$ and δ is the dimension of level $k \geq 1$.

Let $\{l_n, n \geq 0\}$ and $\{\bar{l_n}, n \geq 1\}$ be the marginal distribution of the levels of $X(t)$ and $\bar{X(t)}$ respectively, in the long run as the system gets stabilized.

Let $\bar{z} = (z_1, z_2, \cdots)$ be an invariant measure for $\bar{X(t)}$

Define $\bar{L}_n = \bar{z}_n e$ and $P_0^{-1} = \sum_{n=1}^{\infty} \bar{L}_n$.

If $P_0^{-1} < \infty$, then the equilibrium distribution for $\bar{X(t)}$ exists and $\bar{l}_n = P_0 \bar{L}_n$

But the structure of $\bar{X(t)}$ shows that $\{\bar{l_n}, n \geq 1\}$ can be considered as an equillibrium distribution of a standard birth-and-death process on state space $\{i \geq 1\}$ with trasition rates $\bar{q}(i,j)$ given by

$$\begin{aligned}\bar{q}(0,1) &\geq 0\\ \bar{q}(i,i+1) &= (A_1e)_{max}, \quad i \geq 1\\ \bar{q}(1,0) &= 0\\ \bar{q}(i,i-1) &= (C_{i-1}e)_{min}, \quad i \geq 2\end{aligned}$$

So $\{\bar{l}_n, n \geq 1\}$ is given by

$$\bar{l}_n = P_0 \prod_{i=1}^{n-1} \frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)} \tag{5}$$

This equation (5) shows that a sufficient condition for $P_0^{-1} < \infty$ is that $\frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)} \leq r < 1$ for all $i \geq N$ for some N.

Thus if $\{\bar{l}_n, n \geq 1\}$ exists, the steady state distribution $\bar{x}$ of $\bar{X(t)}$ must exist and therefore x must exist. Since $\bar{X(t)}$ stochastically dominates $X(t)$, we have

$$\sum_{n=K^*}^{\infty} l_n \leq \sum_{n=K^*}^{\infty} \bar{l}_n$$

and hence it is sufficient to fix K^* such that

$$\sum_{n=K^*}^{\infty} \bar{l}_n < \epsilon$$

We use the K^* obtained by the above method to fix the truncation level and employ the Nueuts-Rao procedure in numerical computations. Thus $x_k(K^*)$, $1 \leq k \leq K^*$, is given by

$$x_k(K^*) = x_0(K^*) \prod_{l=0}^{k-1} R_l$$

where $x_0(K^*)$ satisfies

$$x_0(B_0 + R_0C_1) = 0$$

The components of x above the level K^* are given by $x_{K^*+i} = x_{K^*} \prod R_{K^*+j-1}$ and equation (4) becomes

$xe = x_{K^*+1}(I - R_{K^*})^{-1}e + x_0(K^*)e + x_0(K^*) \sum_{k=1}^{K^*} \prod_{l=0}^{k-1} R_l e = 1$. Note that $x_{K^*+1}(I - R_{K^*})^{-1}e < \epsilon$ for our choice of K^*

Performance Measures

For the evaluation of system performance measures we partition each x_i in the steady state probability vector $x = (x_1, x_2, x_3, \cdots)$ as follows: Let $x_{i+1} = (y_i(0,0,0,0), y_i(0,0,0,1), \cdots, y_i(K,0,2,1))$, $for \quad i \geq 0$ where x_{i+1} is a row vector of order $3(K+1)(K+2)$ and $y_i(a,b,c,d)$ is a row vector corresponds to $N_2(t) = a, N_3(t) = b, N_4(t) = c, N_5(t) = d$, $0 \leq a,b \leq K$, $a+b \leq K$, $c = 0,1,2$, $d = 0,1$

- Average number of ordinary customers in the system (avor).

$$avor = \sum_{i=0}^{\infty} i x_{i+1} e$$

- Average number of priority generated customers in the system (avpr).

$$avpr = \sum_{a=0}^{K}\sum_{b=0}^{K-a}[y_0(a,b,1,0)+2y_0(a,b,1,1)]e + \sum_{i=1}^{\infty}\sum_{a=0}^{K}\sum_{b=0}^{K-a}[y_i(a,b,0,1)+y_i(a,b,1,0)+2y_i(a,b,1,1)]e$$

- Average number of priority generated customers lost per unit time (avprl).

$$avprl = \sum_{i=1}^{\infty}\sum_{a=0}^{K}\sum_{b=0}^{K-a} i\gamma[y_i(a,b,0,1)+y_i(a,b,1,1)+y_i(a,b,2,1)]e$$

- Average number of interrupted customers lost per unit time (avinl).

$$avinl = \sum_{i=1}^{\infty}\theta[y_i(0,K,0,0)+y_i(0,K,0,1)]e$$

- Average number of interrupted customers in buffer B_1 (avinb1).

$$avinb1 = \sum_{i=1}^{\infty}\sum_{a=0}^{K-1}\sum_{b=1}^{K-a}\sum_{c=0}^{2}\sum_{d=0}^{1} by_i(a,b,c,d)e$$

- Average number of interrupted customers in buffer B_2 (avinb2).

$$avinb2 = \sum_{i=1}^{\infty}\sum_{a=1}^{K}\sum_{b=0}^{K-a}\sum_{c=0}^{2}\sum_{d=0}^{1} ay_i(a,b,c,d)e$$

- Probability that the server is idle (pidle).

$$pidle = x_1 e$$

- Probability that the server is busy with an ordinary customer (psbor).

$$psbor = \sum_{i=1}^{\infty}\sum_{a=0}^{K}\sum_{b=0}^{K-a}\sum_{d=0}^{1} y_i(a,b,0,d)e$$

- Probability that the server is busy with a priority generated customer (psbpr).

$$psbpr = \sum_{i=0}^{\infty}\sum_{a=0}^{K}\sum_{b=0}^{K-a}[y_i(a,b,1,0)+y_i(a,b,1,1)]e$$

- Probability that the server is busy with a B_2 customer (psbb2).

$$psbb2 = \sum_{i=1}^{\infty}\sum_{a=0}^{K}\sum_{b=0}^{K-a}[y_i(a,b,2,0)+y_i(a,b,2,1)]e$$

- Probability that an interrupted customer is lost (pinl).

$$pinl = \frac{\theta}{\theta+\mu_2}\sum_{i=0}^{\infty}\sum_{c=0}^{1}\sum_{d=0}^{1} y_i(0,K,c,d)e$$

Cost Analysis

We calculate the optimal buffer size K by defining the expected total cost by

$ETC = K \times c_h \times avinb1 + c_l \times avinl$

where, c_h=Holding cost of one customer in buffer in B_1,

c_l=The loss of the system due to one customer if B_1 is full.

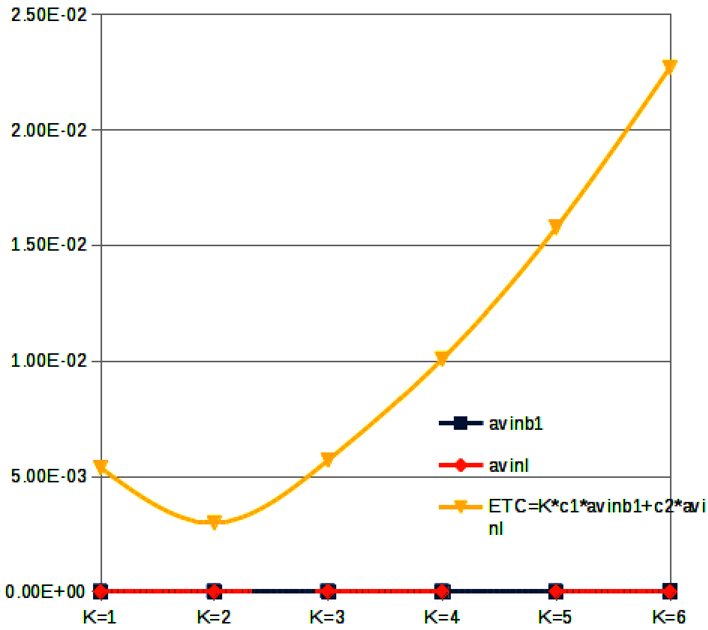
Numerical Example

Let $\lambda = 1, \mu_0 = 0.5, \mu_1 = 1.0, \mu_2 = 1.5, \theta = 1.5, \eta = 0.5, \gamma = 10.0, c_h = \$500, c_l = \$2500.$

Table of ETC with different buffer sizes.

K	$avinb1$	$avinl$	$ETC = K.c_1.avinl + c_2.avinl$
1	1.26×10^{-6}	1.90×10^{-6}	5.37×10^{-3}
2	2.52×10^{-6}	1.87×10^{-7}	2.98×10^{-3}
3	3.78×10^{-6}	1.34×10^{-8}	5.70×10^{-3}
4	5.04×10^{-6}	7.38×10^{-10}	1.01×10^{-2}
5	6.31×10^{-6}	3.25×10^{-11}	1.58×10^{-2}
6	7.57×10^{-6}	1.19×10^{-12}	2.27×10^{-2}

Graph of ETC with different buffer sizes.



Conclusion

A single server customer induced interrupted queueing system with self generation of priority is analyzed in this paper. Arrival of customers are according to Poisson Process and service rates are exponential. The interruption we discussed here is Customer induced interruption. Performance measures required for an appropriate system designing were computed and a cost function was numerically analyzed.

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