Numerical Methods Using Matlab

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Abstract: Numerical Analysis is a branch of mathematics which deals with the study of approximation techniques for solving problems by taking into account the possible errors, which may occur due to our approximation. Numerical methods is a set of procedures done to find the solution of a problem together with error estimates. In this paper we deal with the numerical methods used in finding solutions of differential equation. Here we consider first order differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$. And most importantly we have used the software MATLAB (Matrices Laboratory). It is a fourth generation programming language and numerical analysis environment. By analyzing numerical methods using in finding solutions of differential equation we arrive at a conclusion by finding out the most efficient method among the ones considered.

Index Terms **- Numerical analysis, Numerical Methods, Differential equation, MATLAB.**

1. Introduction

Numerical method is a set of procedures done to find the solution of a problem together with error estimates. This enables us to find the solution of complex problems with numerous steps but using simple operations. Algorithms, computational steps or flow charts are provided for some of the numerical methods and these can be easily be transformed into a computer program by including suitable input/output statements.

One of the main reasons for introducing numerical methods is that it is not always possible to find as analytical solution of a problem as it may be really time consuming. Also it is not possible to solve highly nonlinear equations using analytical techniques. In all these we can apply numerical methods to find an approximate solution for our problem.

In this paper we deal with numerical methods used in finding solutions of differential equations. Here we consider the first order differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$. We discuss some basic methods like Taylor's series method, Picard's method and introduce the methods such as Euler's method, Modified Euler's method and R-K method to solve I.V.P. We also introduce predictor method – Milne method to solve I.VP's.

And most importantly we have used the software MATLAB (Matrices Laboratory). It is a well known that the programming effort is considerably reduced by using standard functions and subroutines. We also introduce the basic concepts of perturbation equations using numerical methods and MATLAB and solved Airy's equation using power series method and plot its graph using with the help of MATLAB.

2. Basic Methods

We consider numerical methods for solving ordinary equations, i.e. those differential equations that have only one independent variable. We shall consider some of the methods commonly used to solve differential equations.

2.1 Taylor's Series Method

Consider the first order differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

We can expand $y(x)$ as a power series in $(x - x_0)$ in the nbd of x_0 by Taylor's series.

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \ldots$$

If $x = x_1 = x_0 + h$ and y can be expanded as Taylor's series about $x = x_1$ and we have

$$y(x_1 + h) = y(x_2) = y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \ldots$$

Continuing this way we find

$$y_{r+1} = y_r + \frac{h}{1!}y'_r + \frac{h}{2!}y''_r + \ldots \text{ Where } y_{r+1} = y(x_{r+1}); r = 1,2,3,\ldots$$

The solution $y(x)$ is given as a sequence $y_0, y_1, y_2, \ldots$

2.1.1 EXAMPLE

Suppose $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$

We find y', y'', y''' at $(x_0, y_0) = (0,1)$

Substituting $r = 0$ in Taylor's series expansion, we can find $(x_1, y_1) = (0.1, 0.9003)$

Proceeding like this we can find (x_2, y_2)

2.2 PICARDS METHOD

Consider the differential equation, $\frac{dy}{dx} = f(x, y)$. Picard method involves integration while Taylor series method involves differentiation of the function f. We obtain $y_1, y_2, y_3, y_4, y_5, ...$ where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})$$

2.2.1 EXAMPLE

Suppose $\frac{dy}{dx} = 3x + y^2$ with initial condition y=1 when x=0.

The first approximation is $y_1 = 1 + \int_0^x f(x, y_0)dx = 1 + x + 3\frac{x^2}{2}$

The second approximation is $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx = 1 + x + 5\frac{x^2}{2} + 4\frac{x^3}{3} + 3\frac{x^4}{4} + \frac{9}{20}$

The third approximation involves squares of y_2 which is a big expression. So we stop with y_2.

When $x = 0.1, y = 1.1264$

When $x = 0.2, y = 1.3120$

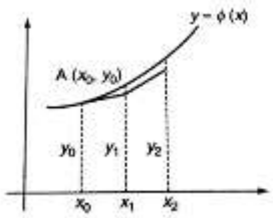
3. COMPARISON OF SOLUTIONS OF O.D.E'S USING MATLAB:

3.1 EULER'S METHOD

Euler's method is one of the simplest and oldest method of finding numerical solution for differential equation. It is a step by step method.

Consider the differential equation, $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. In Euler's method we approximate a curve in a small interval by a straight line.

Let $y = \phi(x)$ be the curve representing the actual solution. Solving using equation of the tangent at (x_0, y_0), finally we get the general formula, $y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2,3,...$ This is called Euler's algorithm.



3.2 EULER'S METHOD - MATLAB

```
%eulers method
clear all
clc
a=0;
b=1;
y(1)=1;
n=10;
t(1)=a;
h=(b-a)/n;
```

```
f=@(y,t)(y+y);
for i=1:(n+1)
y(i+1)=y(i)+h*f(y(i),t(i));
t(i+1)=a+i*h;
fprintf('when t(i+1)=%f y(i+1)=%f\n',t(i),y(i))
end
plot(t,y,'-r','LineWidth',4)
hold on
plot(t,-t-1+2*exp(t),'-b','LineWidth',4)
hold off
```

3.2.1 EXAMPLE

Consider the differential equation, $\frac{dy}{dx} = x + y$ and $y_0 = 1$. We will find the approximate value of y corresponding to the point x=1. For this we take n=10 and h=0.1. The various calculations are showed in table1.

Table 1

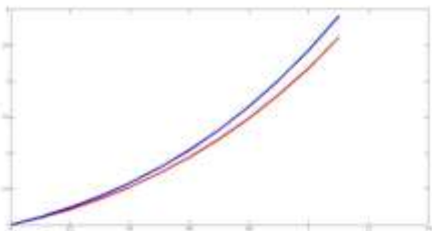
x	y	$x+y=\frac{dy}{dx}$	New y = Old $y + 0.1 \times \frac{dy}{dx}$
0.0	1.00	1.00	$1.00+0.1\times(1.00)=1.10$
0.1	1.10	1.20	$1.10+0.1\times(1.20)=1.22$
0.2	1.22	1.42	$1.22+0.1\times(1.42)=1.36$
0.3	1.36	1.66	$1.36+0.1\times(1.66)=1.53$
0.4	1.53	1.93	$1.53+0.1\times(1.93)=1.72$
0.5	1.72	2.22	$1.72+0.1\times(2.22)=1.94$
0.6	1.94	2.54	$1.94+0.1\times(2.54)=2.19$
0.7	2.19	2.89	$2.19+0.1\times(2.89)=2.48$
0.8	2.48	3.29	$2.48+0.1\times(3.29)=2.81$
0.9	2.81	3.71	$2.81+0.1\times(3.71)=3.18$
1.00	3.18		

Thus the appropriate values of y at x=1 is 3.18. The actual solution of this differential equation is $y = -x - 1 + 2\exp^x$. The comparison of actual method and Euler's method results are shown in table 2.

Table 2

x	y (actual method)	y (Euler's method)
0.0	1.00	1.00
0.1	1.11	1.10
0.2	1.24	1.22
0.3	1.39	1.36
0.4	1.58	1.53
0.5	1.79	1.72
0.6	2.04	1.94
0.7	2.32	2.19
0.8	2.65	2.48
0.9	3.01	2.81
1.00	3.43	3.18

The blue one represents the graph of original solution and red one represents the graph of Euler's method solution. As we increase the value of n, the accuracy of the solution will increase.



3.3 MODIFIED EULER'S METHOD

We obtain the modified Euler's method by replacing the Euler's Method like,

$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y_{n+1})\}$$

3.4 MODIFIED EULER'S METHOD– MATLAB

```
clc
```

```
clear all
% initial condition
f=inline('x+y');
 n = input('Enter the value of n:');

xn=input('enter the value of xn:');
x(1)=0;
y(1)=1;
h=input('enter the value of h:');
for i=1:n
y(i+1) = y(i)+h*f(x(i)+0.5*h,y(i)+0.5*h*f(x(i),y(i)));
x(i+1) = x(i)+h;
fprintf('when x(i+1)=$\%$f y(i+1)=\%6.4f$\setminus$n',x(i),y(i))
end
plot(x,-x-1+2*exp(x),'-m','LineWidth',4)
hold on
plot(x,y,'+b','LineWidth',7)
hold off
```

3.4.1 EXAMPLE

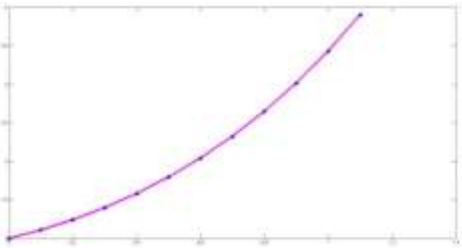
Consider the equation, $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$

The values of y calculated by Modified Euler's Method at the point 0.1,0.2,0.3 and the values got by actual method are shown in table 3.

Table 3

x	y(actual value)	y(Modified Euler's method)
0	1.0000	1.0000
0.1	1.1105	1.1100
0.2	1.2428	1.2421
0.3	1.3997	1.3985
0.4	1.5836	1.5818
0.5	1.7974	1.7949
0.6	2.0448	2.0409
0.7	2.3275	2.3231
0.8	2.6510	2.6456
0.9	3.0192	3.0124
1.0	3.4365	3.4282

The magenta line represents the original solution and the blue dots represent the solution got by Modified Euler's method. So we can conclude that Modified Euler's methpd gives more accuracy as well as time consumption over Euler's Method.



3.5 RUNGA – KUTTA METHOD (R-K METHOD)

Two famous German mathematicians namely Runga and Kutta developed an algorithm named R – K method to solve differential equation in another way. The advantage of this method is that it requires only values of the function at some specified points.

3.5.1 FIRST ORDER RK METHOD

Euler's Method is the first order term in R-K method.

$$y_1 = y_0 + \frac{h}{2} f(x_0, y_0)$$

3.5.2 SECOND ORDER RK METHOD

Modified Euler's method gives the second term in R-K method.

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) \text{ where } k_1 = hf(x_0, y_0) \text{ and } k_2 = hf(x_0 + h, y_0 + h)$$

3.5.3 THIRD ORDER RK METHOD

The third order R-K formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3);$$

where, $k_1 = hf(x_0, y_0)$, $k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}h)$, $k_3 = hf(x_0 + h, y_0 + k')$; $k' = hf(x_0 + h, y_0 + k_1)$

3.6 RK METHOD – MATLAB

```
clc
clear all
% initial condition
f=inline('x+y');
x(1) = 0;
y(1) = 1;
h=input('enter the value of h:');
xn=input('enter the value of xn:');
n=input('enter the value of n:')
for i = 1:n
k1=h*f(x(i),y(i));
k2=h*f(x(i)+h/2,y(i)+k1/2);
k3=h*f(x(i)+h/2,y(i)+k2/2);
k4=h*f(x(i)+h,y(i)+k3);
 y(i+1) = y(i)+1/6*(k1+2*(k2+k3)+k4);
x(i+1) = x(i)+h;
fprintf('when x(i+1)=$\%$f y(i+1)=\%6.4f$\setminus$n',x(i),y(i))
end
plot(x,-x-1+2*exp(x),'-r','LineWidth',3)
hold on
plot(x,y,'+-b','LineWidth',5)
hold off
```

3.6.1 Example

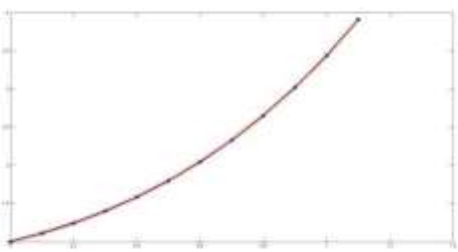
Consider the equation, $\frac{dy}{dx} = x + y; y(0) = 1$

On solving we get $k_1 = 0.2000, k_2 = 0.2400, k_3 = 0.2440, k_4 = 0.2888$. Substituting these values, we get the fourth R-K formula as $y_1 = 0.2428$

Table 4

x	y(actual value)	y(R K method)
0.1	1.1105	1.1103
0.2	1.2428	1.2428
0.3	1.399	1.3997
0.4	1.5836	1.5835
0.5	1.7974	1.7969

The red line represents the original solution and the blue dots represent the solution by R-K method. One of the merit of this method is that the operation is identical whether the differential equation is linear or non linear. Thus we can conclude that R-K method have better performance over other methods.



4. Predictor Corrector Method

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. Divide the range of x into a number of subintervals of equal width h. If x_i and x_{i+1} are consecutive points then $x_{i+1} = x_i + h$.

By Euler's method we have, $y_{i+1} = y_i + hf(x_i, y_i), i = 1,2,3,...$ This formula is called predictor formula.

By Modified Euler's method we have, $y_{i+1} = y_i + \frac{h}{2}\{f(x_i, y_i) + f(x_{i+1}, y_{i+1})\}$. This formula is called corrector formula.

The method of refining an initially rough estimate by means of more accurate formula is called predictor method.

4.1 MILNE'S METHOD

It is a predictor-corrector method for solution of ordinary differential equation.

$y_{n+1} = y_{n-3} + 4\frac{h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n), n = 3,4,...$ This is called Milne's Predictor's Formula.

$y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_2 + y'_{n+1})$. This is called Milne's Corrector Formula.

To apply Milne's method, we need four starting values of y. Hence this method is a multi-step method.

4.2 MILNE METHOD – MATLAB

```
function[]=milne()
clc
clear all
format compact
format short g
global x y;
global h;
x=[0 0 0 0 0];
y=[0 0 0 0 0];
x(1)=input('enter the value of x0:');
xr=input('enter the last value of x:');
h=input('enter the spacing value:');
aerr=input('enter the allowed error:');
y=input('enter the value of y(i),i=0,3:');
for i = 1:3
x(i+1)=x(1)+i*h;
x(2:3)= x(2:3)+x(1,1)*6;
end
disp('x predicted corrected');
disp('x y f y f')\
while(1)
if (x(4)==xr)
return
end
x(5)=x(4)+h;
y(5)=y(1)+(4*h/3)*(2*(f(2)+f(4))-f(3));
fprintf('\%f \%f \%f \n',x(5),y(5),f(5));
correct();
while(1)
yc=y(5);
correct();
if(abs(yc-y(5)=aerr))
break;
end
end
for  i=1:4
x(i)=x(i+1);
y(i)=y(i+1);
end
end
function[z]=f(i)
z=x(i)+y(i);
end
function[]=correct()
y(5)=y(3)+(h/3)*(f(3)+4*f(4)+f(5));
fprintf('\%f \%f \n',y(5),f(5))
end
end
```

4.2.1 EXMAPLE

Consider $\frac{dy}{dx} = x + y$ with initial condition y=1 at x=0. By Picard's method we find the first, second and third approximation. When x=0.2, y=1.2200, when x=.04, y=1.5707, when x=0.6, y=2.0374. Now using Milne method(Matlab) with h=0.2 we get at x=0.8, y=2.6343 and at x=1, y=3.4287.

Table 5

x	y(Milne method)	y(actual method)
0.8	2.6343	2.6510
1.0	3.4287	3.4365

5. AIRY'S EQUATION

Airy equation or the Stokes equation is the simplest second order linear differential equation with a turning point. We define a general Airy's equation as $y'' \pm k^2 xy = 0$. In the physical sciences, the Airy function (or Airy function of the first kind) Ai(x) is a special function named after the British astronomer George Biddell Airy (1801-92). The function Ai(x) and the related function Bi(x), called the Airy function of the second kind are sometimes referred to as the Bairy function, are linearly independent solutions to the differential equation.

The Airy function is the solution to Schrödinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field. The Airy function is also important in microscopy and astronomy; it describes the pattern, due to diffraction and interference, produced by a point source of light.

5.1 SOLUTION OF AIRY'S EQUATION USING POWER SERIES

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{x^3 k}{(2.3)(5.6)...((3k-1).3k)} \quad \text{and} \quad y_2(x) = [x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3.4)(6.7)...(3k.(3k+1))}]$$

The above equation forms a fundamental system of solutions for Airy's equation.

5.2 AIRY'S EQUATION AND MATLAB

Now as we seen that airy's equation is important in physical sciences, Matlab has inbuilt airy's equation solution. Also by solving airy's equation in Matlab using dsolve command :

```
syms y(x)
dsolve('D2y-x*y=0','x')
```

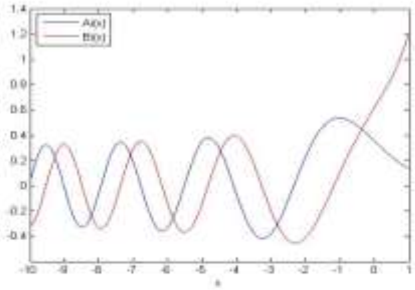
we get the output as C2*airy(0, x) + C3*airy(2, x) ,where airy(0,x) is the airy's function of first kind and airy(2,x) is the airy's fuction of second kind.

Also we can get the values these 2 airy's function .For example if we want the value of airy's function of 1st kind and 2nd kind at the point say 1.5 say, then can we use the Matlab command airy(0,1.5) and airy(2,1.5). Also we can the values of the derivatives of airy's function of two kinds in Matlab.The derivative of airy's function of first kind is represented by airy(1,x) and derivative of airy's function of second kind is represented by airy(3,x).

Now we can plot the solution of airy's equation easily in Matlab. The Matlab command for obtaining the graph of this is as follows:

```
x = -10:0.01:1;
%Calculate Ai(x)
ai = airy(x);
%Calculate Bi(x) using k = 2.
 bi = airy(2,x);
%Plot both results together on the same axes.
figure
plot(x,ai,'-b',x,bi,'-r')
axis([-10 1 -0.6 1.4])
xlabel('x')
legend('Ai(x)','Bi(x)','Location','NorthWest')
```

The graph obtained is as follows:



6. PERTURBED EQUATIONS

Perturbation means a small disturbance in a physical system. Numerical Analysis is the main method for solving perturbation problem. Perturbation method is a method for obtaining approximate solution to algebraic and differential equations for which exact solution is not easy to find. Ɛ is known as the perturbation parameter.

Depending upon the nature of perturbation, a perturbed problem can be divided into two categories.

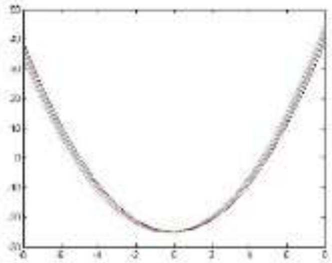
1. Regular Perturbation
2. Singular Perturbation

6.1 REGULARLY PERTURBED EQUATIONS

A perturbation equation is said to be regular if the degree of the perturbed and unperturbed remains the same when $\varepsilon = 0$

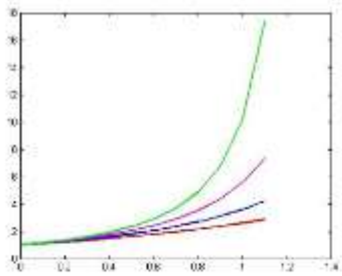
6.1.1 EXAMPLE 1

Consider the quadratic equation $x^2 + \varepsilon x - 25 = 0$. When $\varepsilon = 0$, the equation becomes $x^2 - 25 = 0$. Now we use Matlab for plotting the graph of this for different values of ε. The graph below shows the graph of the quadratic equation for Ɛ=0(blue), Ɛ=0.25(red), Ɛ=0.5(magenta) and Ɛ=0.75(green).



6.1.2 EXAMPLE 2

Consider the differential equation $\frac{dy}{dx} = y + \varepsilon y^2$ with y(0)=1. When Ɛ=0, the equation becomes $\frac{dy}{dx} = y$. Now we solve the perturbed equation for different values of Ɛ using Euler's method. And we can plot them using Matlab. The graph for Ɛ=0(red), Ɛ=0.2(blue), Ɛ=0.4(magenta) and Ɛ=0.6(green) is as shown below.

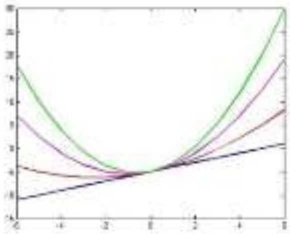


6.2 SINGULARLY PERTURBED EQUATIONS

A perturbed equation is said singularly perturbed if the degree of equation is reduced when Ɛ=0.

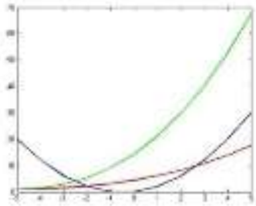
6.2.1 EXAMPLE 1

Consider the quadratic equation $\varepsilon x^2 + x - 5 = 0$. When Ɛ=0 the equation becomes x-5=0. Now we use Matlab for plotting the graph of this for different values of Ɛ. The graph below shows the graph of the quadratic equation for Ɛ=0(blue), Ɛ=0.2(red), Ɛ=0.5(magenta), and Ɛ=0.8(green).



6.2.2 EXAMPLE 2

Consider the differential equation $\varepsilon \frac{dy}{dx} = y + y^2$. When ε=0, the nature of the equation itself changes. It becomes a quadratic equation. Now we solve the perturbed equation for different values of ε using R-K method and plot them using Matlab. The graph for ε=0(blue), ε=0.2(red), ε=0.8(green) is shown below.



Thus with the help of numerical methods and Matlab we can study perturbation equations and see the variations that happen due to the change the value of the parameter ε.

7. CONCLUSION AND FUTURE WORK

In this paper, we have discussed different numerical methods for finding the solution of ordinary differential equations with initial conditions. We can see that Modified Euler's method and R K Method gives the best solution and R K Method is the most widely used method among all these methods. So, we can conclude that these numerical methods are useful for finding an approximate solution of differential equations. We have also discussed an important equation in physical science , Airy's equation and plot its graph with the help of Matlab. Also, we have discussed the some basics concepts of perturbation equations with the help of numerical method and Matlab. So we can conclude that Matlab is very helpful for plotting graphs and reduces our work and time.

This work can be extended in various directions:

The problems we solve here are first order differential equations. We can extend this work by discussing numerical methods for higher order differential equations and program them in Matlab which will be very useful in reducing our time. We have only introduced the basic concepts of perturbed equations and we can extend this by solving more complex perturbed equations by constructing new numerical methods or modifying existing numerical methods and designing suitable Matlab programs for these methods.

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