INTRODUCTION AND APPLICATION OF BANACH-STEINHAUS THEOREM IN 2-BANACH SPACES

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ABSTRACT- A White introduced the notion of cauchy sequences in 2- normed spaces. After he also defined 2-Banach spaces during this he introduced the notion of 2-functional and the norm of 2 functional and proved a remarkable theorem MX [b] C L X L where L is a 2- Banach space and M and [b] are linear manifolds in L [b] being generated by b which is similar to the Hahn-Banach theorem. And he applied his theorem to obtain some result.

KEYWORD- 2-Banach theorem, 2-normed spaces,etc

INTRODUCTION- Let L be a linear space, The pair $(L,\|.,.\|)$ is called a linear 2-normed space provided $\|.,.\|$ satisfies that following condition where $\|.,.\|$ is real valued function defined on $L: a, b.c \in L$

1. $\|a,b\| = 0$ if and only if a and b are linearly dependent,
2. $\| a, b \| = \| b, a \|$,
3. $\| a, \alpha\beta \| = |\beta| \| a, b \| \beta$ real,
4. $\| a, b + c \| \le \| a, b \| + \| a, c \|$.

$\|.,.\|$ is called a 2- norm which has been shown in [1] on to be non negative. A linear 2-normed $(L,\|.,.\|)$ will simply be denoted by L, unless otherwise stated.

A sequence $\{x_n\}$ is L is called a Cauchy sequence if there exist y, $z \in L$ such that y and z are linearly independent and $\lim \|x_n - x_m, y\| = 0$, $\lim \|x_n - x_m, z\| = 0$, A sequence $\{x_n\}$ is called convergent if there is an $x \in L$ such that $\lim \|x_n - x_m, y\| = 0$ for all $y \in L$. In this case we say that $\{x_n\}$ converges to x, write $x_n \to x$, and call x the limit of $\{x_n\}$. A linear 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach spaces.

A 2-functional is real-valued mapping with domain A × C where A and C are linear manifolds of L. Let F be a 2-functional with domain A × C. F is called a linear 2-functional if :

1. $F(a+c, b+d) = F(a,b) + F(a,d) + F(c,b) + F(c,d)$,
2. $F(\alpha a, \beta b) = \beta b F(a,b)$ where α, β are scalars.

Let F be a 2- functional with domain D(F). F is called bounded if there is a constant $K \ge 0$ such that $|F(a,b)| \le K \| a, b \|$ for all $(a,b) \in D(F)$. If F is bounded then the norm of $F \| F \|$, is given by

$\| F \| = \text{glb}\{K : | F(a,b) | \le K \| a, b \| \text{ for all } (a,b) \in D(F)\}$ if F is not bounded, then $\| F \| = +\infty$

Clearly the domain of definition of F may in some cases be L × L.

Theorem. Let F be a bounded linear 2-functional with domain D(F). Then

$\| F \| = \sup\{| F(x,y) | : \| x, y \| = 1, (x,y) \in D(F)\}$

$$=\sup\left\{\frac{|F(x,y)|}{\|(,y)\|}:\|x,y\|\neq 0,(x,y)\in D(F)\right\}.$$

We have a theorem on $M\times[b]\subset L\times L$ where L is a 2-Banach space and M and [b] are linear manifolds L, [b] being generated by the element b, which is similar to the Hahn-Banach theorem.

Definition: Let $\{x_n\}$ be an infinite sequence of elements in L. The series $\sum_{n=1}^{\infty} x_n$ is said to be convergent in L if the sequence of partial sums $\{S_n\}$ where $S_n = x_1 + x_2 + ... + x_n$ is convergent in L. If $S_n \to \infty \text{ as } n \to \infty$, we write $\sum_{n=1}^{\infty} x_n = S.$

Definition: Let L be dimension ≥ 2 and a, b be two linearly independent elements in L. Then L is said to have the property (P) with respect to a and b if $|x,\lambda\|\leq\|x,a+b\|$ for all $x\in L.$ and where $\lambda = a \text{ or } b.$

Theorem : Let L be a 2-Banach space of dimension ≥ 2 and let a and b be two linearly independent elements in L. Suppose L has the property (P) with respect to a and b. Let $\{F_i\}_{i\varepsilon A}$, A is an index set, be a family of bounded linear 2-functionals with domain $Lx[a+b]$ such that $\{F_i(x,a+b)\}_{i\varepsilon A}$ is bounded for each $x\in L$ then sup $\|f_i\|<\infty.$

Proof: Suppose $\{\|F_i\|\}_{ieA}$ is unbounded. We will construct a sequence $\{x_n, a+b\}\subset Lx[a+b]$ and a sequence $\{F_n\}$ from $\{F_i\}_{ieA}$ so that

$$x=\sum_{n=1}^{\infty} x_i \in L \quad \text{and} \quad |F_n(x,a+b)|>n. \tag{1}$$

Since $\{\|F_i\|\}$ is unbounded, there exists $F_1\in\{F_i\}_{ieA}$ with $\|F_1\|>4.$ Hence it follows that

$$\|F_1\|=\sup\left\{\frac{|F_1(x,\alpha(a+b))|}{\|x,\alpha(a+b)\|},\|x,\alpha(a+b)\|\neq 0\right\}$$

$$=\sup\left\{\frac{|F_1(x,(a+b))|}{\|x,(a+b)\|},\|x,(a+b)\|\neq 0\right\}.$$

So there exists $(x',a+b)$ with $\|x',a+b\|=0$ such that $|F_i(x',a+b)|>4.\|x',a+b\|.$ Let

$$x_1=\frac{x'}{4\|x',\alpha(a+b)\|}.$$

Then $x_1\in L, \|x_1,a+b\|=\frac{1}{4}$ and $|F_1(x_1,a+b)|>1.$ Suppose in this way it has been possible to select the elements $x_2,x_3,x_4,....x_{n-1}$ from L and $F_2,F_3,F_4,....F_{n-1}$ from $\{F_i\}_{ieA}$ which satisfy (1), for these n's. Let

$$M_{n-1}=\sup_{i\in A}|F_i(x_1+x_2+...+x_{n-1},a+b)|.$$

Then from hypothesis, M_{n-1} is finite. There exists and F_n, say belonging to $\{F_i\}$ with

$$\|F_n\|>3\cdot 4^n[M_{n-1}+n] \tag{2}$$

we obtain an $x''\in L$, similarly as x' was obtained, such that

$$\frac{|F_n(x'',a+b)|}{\|x'',a+b\|}>\frac{2}{3}\|F_n\|.$$

Let $x_n = \dfrac{x''}{4^n \|x'', a+b\|}$

Then $x_n \in L, \|x_n, a+b\| = \dfrac{1}{4^n}$ and $|F_n(x_n, a+b)| > \dfrac{2}{3}\|F_n\|\dfrac{1}{4^n}$

i.e. $\|F_n\| < \dfrac{3}{2} \cdot 4^n |F_n(x_n, a+b)|.$ (3)

Further $|F_n(x_n, a+b)| > \dfrac{2}{3} \cdot 3 \cdot \dfrac{4^n}{4^n}[M_{n-1} + n],$ by (2)

$= 2[M_{n-1} + n].$ (4)

We form the infinite series

$$x_1 + x_2 + x_3 + \ldots + x_n + \ldots$$

Which we show first to be convergent Let $S_n = x_1 + x_2 + \ldots + x_n, n = 1, 2, \ldots$ and n > m. Using the property (P) we see that

$$\| S_n - S_m, a \| \leq \sum_{i=m+1}^{n} \| x_i, a \| \leq \sum_{i=m+1}^{n} \| x_i, (a+b) \|$$

and similarly

$$\| S_n - S_m, b \| \leq \sum_{i=m+1}^{n} \| x_i, (a+b) \|$$

Since $\| x_n, (a+b) \| = \dfrac{1}{4^n}\ n = 1, 2, \ldots,$ the sequence $\{S_n\}$ is Cauchy. So $\{S_n\}$ converges to an element *x*, say of L i.e. $\sum_{n=1}^{\infty} x_n = x \in L$ Now

$$| F_n(x_{n+1} + x_{n+2} + \ldots, a+b) | \leq \| F_n \{ \| x_{n+1}, a+b \| + \| x_{n+2}, a+b \| + \ldots \}$$

$$= \| F_n \| \left\{ \frac{1}{4^{n+1}} + \frac{1}{4^{n+2}} + \ldots \right\}$$

$$= \| F_n \| \frac{1}{4^n} \cdot \frac{1}{3}. \quad (5)$$

From (3) and (5) we obtain

$$| F_n(x_{n+1} + x_{n+2} + \ldots, a+b) | \leq \frac{1}{2} F_n(x, a+b). \quad (6)$$

Now, $| F_n(x, a+b) | = | F_n(x_1 + x_2 + \ldots + x_n + x_{n+1} + \ldots, a+b) |$

$$= | F_n \{ x_n + (x_1 + x_2 + \ldots + x_{n-1}) + x_{n+1} + \ldots, a+b \} |$$

$$\geq | F_n(x_n, a+b) | - | F_n(x_1 + x_2 + \ldots x_{n-1}, a+b) |$$

$$- | F_n(x_{n+1} + x_{n+2} + \ldots, a+b) |$$

$$\geq |F_n(x_n, a+b)| - \frac{1}{2}|F_n(x_n, a+b)|$$

$$-|F_n(x_1 + x_2 + \ldots + x_{n-1}, a+b)|,$$ by (6)

$$= \frac{1}{2}|F_n(x_n, a+b)| - |F_n(x_1 + x_2 + \ldots + x_{n-1}, a+b)|.$$

Using (4) and the definition of M_{n-1}, it follows that

$$|F_n(x, a+b)| \geq \frac{1}{2}|(x_n, a+b) - M_{n-1}.$$

$$> 2[M_{n-1} + n]\frac{1}{2} - M_{n-1} = n.$$

This contradiction proves the theorem.

APPLICATION Let L denote the set of all polynomials

$$x(t) = x_n + x_1 t + x_2 t^2 + \ldots + x_n t^n$$

of degree n where $x_i, 0 \leq i \leq n$ are real numbers. Here the positive integer n is not fixed. With the usual definition of addition and scalar (real) multiplication, L is a linear space. We define

$\|x, y\| = 0$ if x and y are linearly dependent and

$$\|x, y\| = \max_i |x_i| \cdot \max_j |x_j|,$$

Where $y(t) = y_0 + y_t + \ldots + y_m t^m \in L.$ It may be easily verified that

$\|x, y\|$ is a 2-norm on L.

Since i = l and j = t are two linearly independent elements of L, the dimension of L is at least two. Let $x \in L$ and let

$$x(t) = x_0 + x_1 t + \ldots + x_n t^n.$$

Then

$$\|x, i\| = \max_k \|x_k\| = \|x, j\| = \|x, i + j\|.$$

This shows that L has the property (P) with respect to I and j. We write a polynomial $x(t) \in L$ of degree N_x in the form $x(t) = \sum_{j=0}^{\infty} x_j t^j$ where $x_i = 0$ for $j > N_x$. If $x \in L$ the we construct a sequence {Fn} of 2-functionals on Lx[1 + t] by

$$F_n(x,y) = (x_0 + x_1 + ... + x_{n-1})\lambda$$

Where $y = \lambda(1+t)$ and λ is real. Let n be fixed and $x, y \in L, u, v \in [1+t]$. Suppose the degree of x be N_x and the degree of y be N_y Then

$x = \sum_{j=0}^{\infty} x_j t^j, x_j = 0$ for $j > N_x$ and $y = \sum_{j=0}^{\infty} y_j t^j, y_j = 0$ for $> N_y$.

Then $x + y = \sum_{j=0}^{\infty} (x_j + y_j) t^j$ where

$x_j + y_j = x_j$ for $N_x > N_y; N_y < j \le N_x$,

$= y_j$ for $N_y > N_x; N_x < j \le N_y$,

$= 0$ for $j > \max\{N_x, N_y\}$.

Also $u = \lambda(1+t)$ and v = u (1 + t) where λ and u are real. It is seen easily that $F_n(x+y, u+v)$ $= F_n(x,u) + F_n(x,v) + F_n(y,u) + F_n(y,v)$ and $F_n(\alpha x, \beta u) = \alpha\beta F_n(x,u)$ where α, β are real numbers. Therefore F_n is linear for each n.

Also $|F_n(x,u) = \left|\sum_{j=0}^{n-1} x_j \lambda\right| \le \sum_{j=0}^{n-1} |x_j \lambda| \le n \max_{0 \le j \le n-1} |x_j \lambda| \le nx$

$x \max_j \| x_j \lambda | = n \| | x, u |$, so that F_n is bounded for each n. If $x \in L$ then x(t) is a polynomial of degree N_x which has at most $N_x + 1$ non-zero co-efficients and therefore $|F_n(x, 1+t)| \le (N_x + 1) \max_j |x_j|$ for each n where $\max_j |x_j|$ is taken over $x_0, x_i, ..., x_N$. Therefore the sequence $\{F_n(x, 1+t)\}$ is bounded for each $x \in L$. On the other hand, if $x(t) = 1 + t + .t^2 + t^n$ then $\| x, u \| = |\lambda|$ and $F_n(x,u)$ $= (1 + 1 + ... + 1)\lambda$. So $F_n(x,u) = n|\lambda| = n\| x, u \|$. So,

$$\| F_n \| \ge \frac{|F_n(x,u)|}{\| x, u \|} = n$$

Which shows that $\{\| F_n \|\}$ is not bounded. Therefore by the above theorem L with the above 2-norm is not a 2-Banach space.

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