MATHEMATICAL MODELING ON GROWTH OF POPULATION WITH HARVESTING: SOME ASPECTS

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ABSTRACT

The growing human need for more energy and more food has been leading to increase exploitation of biological resources. On the other hand, the present global demand is to protect ecology and environment. In the face of these two opposing approaches, we are looking for a sustainable development policy in every sphere of our life. The present paper provides mathematical modeling on growth of population with harvesting: some aspects. In this paper we have discuss about Logistic Model with Constant Harvesting Rate, Growth of Population, Harvesting Rate Being Proportional to Population Size, Growth of Predator-prey Populations with Harvesting and Growth of Populations in Competition under Harvesting.

Introduction

The ability to predict the population size of a group of individuals is extremely useful to the study of ecology. It allows for the estimation of the various effects imposed upon a group by internal and external forces [1]. We note that the word force has a different meaning in population modeling than in physics. You can think of these forces as factors that impact the population – for example, availability of food, spread of disease, interactions with other species. Among the most important concerns in population ecology is the effect of harvesting a natural population [7]. Harvesting can represent reduction of the population due to hunting or capturing individuals, which in effect removes individuals from the population \[6]. A deliberate decision to cut back expenditure of all kinds on a particular product (usually in the decline stage of its life cycle) in order to maximize profit from it, even if in doing so it continues to lose market share[2,3].

This outlook has necessitated scientific management of exploitation of biological resources like fisheries and forests, and in order to achieve this goal, bio-economic modeling is being increasingly used now-a-day [9]. The basic issues related to this field were presented by Colin Clark in his first book [4]. The technique and issues associated with bio-economic modeling were further developed by Clark in his book [5]. An interested reader may find contributions of the author the articles [8, 10].

(A) Logistic-Model with Constant Harvesting Rate

Let x (t) be the population size at time t, and let F(x) be the natural biological growth rate in the absence of harvesting. Also, let h (t) is the harvesting rate. Then we get

$$dx/dt = F(x) - h(t) \tag{1.1}$$

Let us consider the particular case when

$$F(x) = ax - bx^2,\ h(t) = h = \text{constant} \tag{1.2}$$

So that

$$dx/dt = ax - bx^2 - h \quad (1.3)$$

The positions of equilibrium are given by

$$x_1, x_2 = \frac{1}{2b}(a \pm \sqrt{a^2 - 4bh}). \quad (1.4)$$

The maximum value of $ax - bx^2$ is $a^2/(4b)$ and occurs when $x = a/(2b)$. Now three cases arise (Fig. 1.1)

Case I

$h < a^2/(4b)$, i.e., harvesting rate is less than the maximum natural growth rate of population. In this case, there are two points of equilibrium, namely, x_1 and x_2, one less than $a/(2b)$ and the other greater than $a/(2b)$. Equation (1.3) now becomes.

$$dx/dt = -b(x - x_1)(x - x_2) \quad (1.5)$$

So, that dx/dt is negative when $x < x_1$, it is positive when $x_1 < x < x_2$, and it is again negative when $x > x_2$. The direction field is indicated by arrows in Fig. 1.1. It is also obvious from the first component in the figure that x_1 is a point of unstable equilibrium and x_2 is a point of stable equilibrium. If $x(0) < x_1$, then the population tends to extinction, if $x(0) > x_1$, then it approaches x_2.

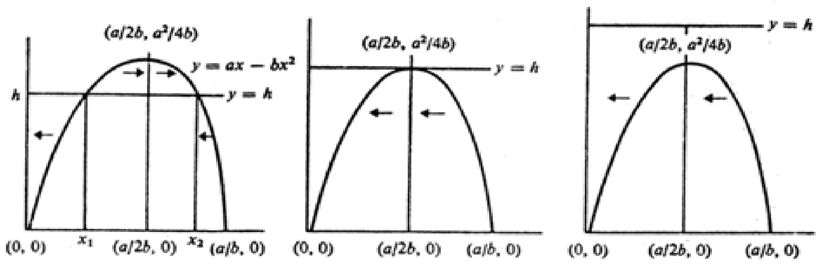

Fig. 1.1: Points of equilibrium for constant harvesting rate.

Integrating (1.5), we obtain

$$\frac{x - x_1}{x - x_2} = \frac{x(0) - x_1}{x(0) - x_2} \exp[(x_2 - x_1)bt] \quad (1.6)$$

If $x(0) < x_1 < x_2$, then the population becomes extinct in time T given by

$$T = \frac{1}{b(x_2 - x_1)} \ln\left(\frac{x_1}{x_2} \frac{x(0) - x_2}{x(0) - x_1} \right) \quad (1.7)$$

If $x(0) > x_1$, then $x \to x_2$ asymptotically as $t \to \infty$, If $x_1 < x(0) < a/(2b)$, then the population growth curve still has a point of inflexion at $x = a/(2b)$, though the final population size is only $x_2 < a/b$.

Case 2

$h = a^2/(4b)$, i.e., the harvesting rate is equal to the maximum natural growth rate of the population. Here only one point of equilibrium exists at $x = \frac{a}{2b}$ which is semi-stable since (1.5) becomes

$$\frac{dx}{dt} = -b\left(x - \frac{a}{2b}\right)^2 \quad (1.8)$$

Also, if x(0) > a/(2b), then x(t) $\rightarrow$a/(2b); on the other hand, if x(0) < a/(2b), then x(t) $\rightarrow$0.

Case 3

h > a^2/ (4b), i.e., the harvesting rate is greater than the maximum natural growth rate of the population. In this case there is no point of equilibrium and, regardless of the value of x (0), x (t) becomes zero after a finite time.

For the generalized logistic laws (i.e., for those laws in which F(x) may not be ax – bx^2, but the graph of F(x) has the same shape, starting from the origin, rising to a maximum, and then falling to zero again), conclusions similar to those found in (i)-(iii) will hold.

(B) Growth of Population, Harvesting Rate Being Proportional to Population Size.

We assume that harvesting rate is proportional to the product of the effort E and to the stock level or the population size x. This is known as catch-per-unit-effort hypothesis. We also choose our units so that the constant of proportionality is unit. Then (1.1) becomes

$$dx/dt = F(x) - Ex. \quad (1.9)$$

For the logistic law, (1.9) gives

$$dx/dt = (a - E)\, x - bx^2 \quad (1.10)$$

So that, if a > E, the growth law is still the logistic law with parameters a – E and b, with the final population size (a – E)/b, and with a point of inflexion at (a – E)/ (2b). Hence, as expected, the limiting population size is reduced due to harvesting.

The ultimate yield Y obtained per unit time is given by

$$Y = E\frac{a-E}{b} \quad (1.11)$$

This is a quadratic function of E. As E increases from 0 to a/b, the yield Y per unit time increases from 0 to a^2/ (4b); but, if E is increased further, Y decreases from a^2/ (4b) to zero.

E can also be prescribed as a function E (t) of time. Then (1.10) gives

$$\frac{1}{x}\frac{dx}{dt} = a - bx - E(t) \quad (1.12)$$

Whose integration should give x as a function of t.

(C) Growth of Predator-prey Populations with Harvesting

We consider two cases, namely, constant harvesting rates and harvesting rates proportional to respective population sizes. We know that basic equations of prey predator model are

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_1, N_1 N_2,\ a_1\alpha_1 > 0$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_2\, N_1 N_2,\ a_2\alpha_2 > 0 \quad (1.13)$$

<u>Case 1</u>

For constant harvesting rates h_1, h_2 of prey and predator species, (1.12) becomes.

$$\frac{dN_1}{dt} = a_1N_1 - \alpha_1N_1N_2 - \mathrm{h}_1, \quad \frac{dN_2}{dt} = -a_2N_2 + \alpha_2N_1N_2 - h_2$$

The points of equilibrium are obtained by solving the equations

$$\bar{N}_1(a_1 - \alpha_1\bar{N}_2) - h_1 = 0 \quad \bar{N}_2(-a_2 + \alpha_2\bar{N}_1) - h_2 = 0 \tag{1.14}$$

i.e., they are obtained as the points of intersection $(\bar{N}_1 > 0, \bar{N}_2 > 0)$ of the rectangular hyperbolas shown in fig. 1.2. Thus

$$\bar{N}_1 > \frac{a_2}{\alpha_1}, \quad \bar{N}_2 < \frac{a_1}{\alpha_1} \tag{1.15}$$

So that harvesting at constant rates increases the equilibrium population of prey and decreases the equilibrium population of predator. The secular equation for discussing the stability of this position is

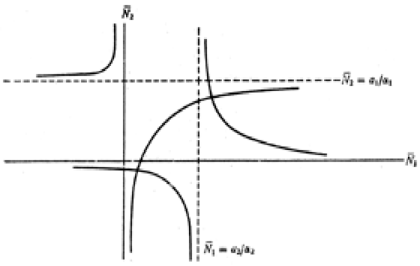

Fig. 1.2: Points of equilibrium for predator-prey model with harvesting

$$\lambda^2 - \lambda(a_1 - \alpha_1\bar{N}_2 - a_2 + \alpha_2\bar{N}_1) + (a_1 - \alpha_1\bar{N}_2)(-a_2 + \alpha_2\bar{N}_1) + \alpha_1\alpha_2\bar{N}_1\bar{N}_2 = 0 \tag{1.16}$$

This is of the form

$$\lambda^2 - A\lambda + B = 0 \quad (A > 0, B > 0) \tag{1.17}$$

So that the equilibrium is unstable. Thus, harvesting at constant rates changes the equilibrium from neutral to unstable. The equilibrium position changes from a centre to an unstable focus or an unstable node.

As $t \to \infty$, the predator species tends to extinction and prey species will ultimately grow according to the law

$$\frac{dN_1}{di} = a_1N_1 - b_1 \tag{1.18}$$

<u>Case 2</u>

If harvesting rates are proportional to population sizes, then

$$\frac{dN_1}{dt} = (a_1 - E_1)N_1 - \alpha_1N_1N_2, \quad \frac{dN_2}{dt} = -(a_2 + E_2)N_2 + \alpha_2N_1N_2 \tag{1.19}$$

The equilibrium position is given by

$$\bar{N}_1 = \frac{a_2 + E_2}{\alpha_2}, \quad \bar{N}_2 = \frac{a_1 - E_1}{\alpha_1} \tag{1.20}$$

so that again harvesting increases the equilibrium population of prey and decreases the equilibrium population of predators (see fig. 1.3). If $a_1 > E_1$, the equilibrium position is still a centre, and conservative oscillations occur about the new equilibrium position.

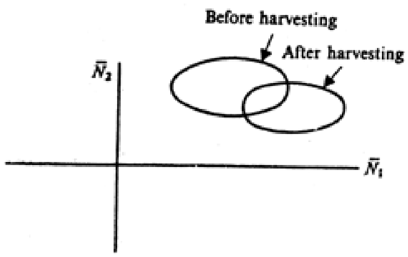

Fig. 1.3: Trajectories before and after harvesting

(D) Growth of Populations in Competition under Harvesting

Here also we consider two cases, one of constant rates of harvesting and the other of rates of harvesting proportional to population sizes.

Case 1

$$\frac{dN_1}{dt} = N_1(a_1 - b_{11}N_1 - b_{12}N_2) - h_1, \quad \frac{dN_2}{dt} = N_2(a_2 - b_{21}N_1 - b_{22}N_2) - h_2 \tag{1.21}$$

Let $(\bar{N}_1, \bar{N}_2), (\bar{N}_1', \bar{N}_2')$ be the respective equilibrium positions without and with harvesting. Then

$$a_1 - b_{11}\bar{N}_1' - b_{12}\bar{N}_2' > 0, \quad a_2 - b_{21}\bar{N}_1' - b_{22}\bar{N}_2' > 0 \tag{1.22}$$

$$a_1 - b_{11}\bar{N}_1 - b_{12}\bar{N}_2 = 0, \quad a_2 - b_{21}\bar{N}_1 - b_{22}\bar{N}_2 = 0 \tag{1.23}$$

$$b_{11}\bar{N}_1 + b_{12}\bar{N}_2 > b_{11}\bar{N}_1' + b_{12}\bar{N}_2', \quad b_{21}\bar{N}_1 + b_{22}\bar{N}_2 > b_{21}\bar{N}_1' + b_{22}\bar{N}_2' \tag{1.24}$$

$$b_{11}(\bar{N}_1 - \bar{N}_1') > b_{12}(\bar{N}_2' - \bar{N}_2), \quad b_{21}(\bar{N}_1 - \bar{N}_1') > b_{22}(\bar{N}_2' - \bar{N}_2) \tag{1.25}$$

$$\mathrm{b}_{12}(\bar{N}_2 - \bar{N}_2') > b_{11}(\bar{N}_1' - \bar{N}_1) \quad b_{22}(\bar{N}_2 - \bar{N}_2') > b_{21}(\bar{N}_1' - \bar{N}_2) \tag{1.26}$$

So that, if the equilibrium population of one species increases, the equilibrium population of the other species decreases.

The secular equations for determining the stability of the new and old positions are given by

$$\left.\begin{aligned} \lambda^2 + \lambda(b_{11}\bar{N}_1' + b_{22}\bar{N}_2') + (b_{11}b_{22} - b_{12}b_{21})\bar{N}_1'\bar{N}_2' = 0 \\ \lambda^2 + \lambda(b_{11}\bar{N}_1 + b_{22}\bar{N}_2) + (b_{11}b_{22} - b_{12}b_{21})\bar{N}_1\bar{N}_2 = 0 \end{aligned}\right\} \tag{1.27}$$

so that the nature of the equilibrium does not change and is stable or unstable according as

$$b_{11}b_{22} > or < b_{12}b_{21} \tag{1.28}$$

If, however, $b_{11} = 0$, $b_{22} = 0$, i.e., if there are no self-limiting terms in (1.21), then (.24) give

$$\bar{N}_1' < \bar{N}_1, \quad \bar{N}_2' < \bar{N}_2 \tag{1.29}$$

So that the equilibrium sizes of both species are reduced.

Case 2

For rates of harvesting proportional to population sizes, we get

$$\frac{dN_1}{dt} = N_1(a_1 - E_1 - b_{11}N_1 - b_{12}N_2) = N_1[\mathrm{a}_1(1-k_1) - b_{11}N_1 - b_{12}N_2] \tag{1.30}$$

$$\frac{dN_2}{dt} = N_2(a_2 - E_2 - b_{21}N_1 - b_{22}N_2) = N_2[a_2(1-k_2) - b_{21}N_1 - b_{22}N_2] \qquad (1.31)$$

We also assume that

$$0 < k_1 < 1, \qquad 0 < k_2 < 1, \qquad (1.32)$$

Thus, by harvesting, a_1 is reduced to $a_1 (1 - k_1)$ and a_2 is reduced to $a_2 (1 - k_2)$. We can now draw the following conclusions based on following figures.

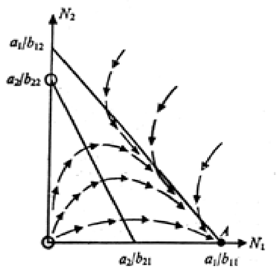

Fig. 1.4 (a): Second species dies

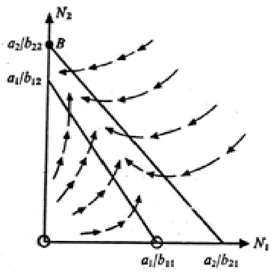

Fig. 1.4 (b): First species dies

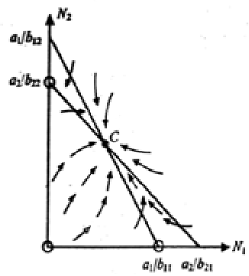

Fig. 1.4 (c): Two species coexist

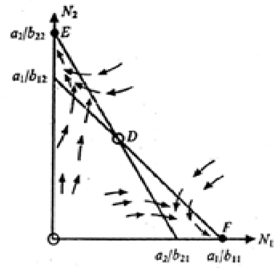

Fig. 1.4 (d): Either species survives, depending on initial conditions

(i) Before harvesting if we have Fig. 1.4(a), then, after harvesting, we can have Fig. 1.4(b) provided.

$$\frac{a_1}{b_{12}} > \frac{a_2}{b_{22}}, \qquad \frac{a_1(1-k_1)}{b_{12}} < \frac{a_2(1-k_2)}{b_{22}}$$

$$\frac{a_1}{b_{11}} > \frac{a_2}{b_{21}}, \qquad \frac{a_1(1-k_1)}{b_{11}} < \frac{a_2(1-k_2)}{b_{21}} \qquad (1.33)$$

Or

$$\frac{1-k_1}{1-k_2} < \frac{a_2 b_{12}}{a_1 b_{22}} < 1 \quad and \quad \frac{1-k_1}{1-k_2} < \frac{a_2 b_{11}}{a_1 b_{21}} < 1 \qquad (1.34)$$

Or

$$\frac{1-k_1}{1-k_2} < \min\left(\frac{a_2 b_{12}}{a_1 b_{22}}, \frac{a_2 b_{11}}{a_1 b_{21}}\right) < 1 \qquad (1.35)$$

We can always choose k_1, k_2 ($k_1 > k_2$) to make (1.35) true.

Thus, before harvesting if the first species survives and the second species tends to extinction, then, by greater harvesting in the first species, we can reverse the position and make the second species survive and the first species disappear. Similarly, before harvesting if the first species tends to disappear, then, by greater harvesting in the second species, we can make the second species disappear and the first species survive.

(ii) Before harvesting if we have Fig. 1.4(a), then, after harvesting, we can have Fig. 1.4(c) provided.

$$\frac{a_2 b_{12}}{a_1 b_{22}} < 1, \quad \frac{a_2 b_{11}}{a_1 b_{21}} < 1,$$

$$\frac{1-k_1}{1-k_2} > \frac{a_2 b_{12}}{a_1 b_{22}}, \quad \frac{1-k_1}{1-k_2} < \frac{a_2 b_{11}}{a_1 b_{21}} \tag{1.36}$$

These inequalities can be possible if

$$\frac{a_2 b_{11}}{a_1 b_{21}} > \frac{a_2 b_{12}}{a_1 b_{22}} \quad or \quad b_{11} b_{22} > b_{12} b_{21} \tag{1.37}$$

which is satisfied since fig. 1.4(c) corresponds to the case when the non-zero equilibrium position is stable.

Thus, before harvesting if the second (first) species tends to disappear, by suitably increasing the rate of harvesting in the first (second) species, we can make the two species coexist in stable equilibrium.

(iii) Before harvesting if we have Fig. 1.4(a), then, after harvesting, we can have Fig. 1.4(d) provided

$$\left.\begin{array}{ll} \dfrac{a_2 b_{11}}{a_1 b_{21}} < 1, & \dfrac{a_2 b_{12}}{a_1 b_{22}} < 1 \\ \dfrac{1-k_1}{1-k_2} < \dfrac{a_2 b_{12}}{a_1 b_{22}}, & \dfrac{1-k_1}{1-k_2} > \dfrac{a_2 b_{11}}{a_1 b_{21}} \end{array}\right\} \tag{1.38}$$

For (1.38) to be true, it is necessary that

$$b_{11} b_{22} < b_{12} b_{21} \tag{1.39}$$

which is satisfied in the case of Fig. 1.4(d). Inequalities (1.38) and (1.39) are mutually exclusive. Thus, Fig. 1.4(a) or 1.4(b) can be changed to Fig. 1.4(c) if (1.38) are satisfied, and it can be changed to Fig. 1.4(d) if (1.39) is satisfied.

Thus, in the absence of harvesting if one of the species tends to disappear and the coexistence position is stable, then, by proper harvesting, we can make the species coexist; but; if the coexistence position is unstable, then, by proper harvesting, we can make the first or second species survive, depending on the initial conditions.

Now let

$$K_1 = \frac{a_2 b_{11}}{a_1 b_{21}}, \quad K_2 = \frac{a_2 b_{12}}{a_1 b_{22}}, \quad K_3 = \frac{1-k_1}{1-k_2} \tag{1.40}$$

Then we obtain the following results:

(i) The second species will disappear before harvesting if

$$K_1 < 1, \; K_2 < 1 \tag{1.41}$$

(ii) The second species will still disappear after harvesting if

$$K_1 > \max(K_1, K_2) \tag{1.42}$$

(iii) The first species will disappear after harvesting if

$$K_3 < \min(K_1, K_2) \tag{1.43}$$

(iv) The two species will coexist after harvesting if

$$0 < K_2 < K_3 < K_1 < 1 \quad (1.44)$$

(v) Either species can disappear after harvesting, depending on the initial conditions if

$$0 < K_1 < K_3 < K_2 < 1 \quad (1.45)$$

Similarly, by adopting suitable harvesting policies, we can discuss the change in any of the four non-degenerate cases in fig. 1.4 (a), 1.4 (b), 1.4 (c) and 1.4 (d) as also the change in the possible degenerate cases.

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