SOME THEOREMS AND RESULTS RELATED TO UNIFORM FUZZY RELATIONS

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Abstract: In this paper we have discussed and studied about uniform fuzzy relations partial fuzzy function, L-Function etc together with some related theorems and corollary.

Key Words: Fuzzy relation, uniform fuzzy relation, fuzzy function, fuzzy homomorphism, automata, L-function.

INTRODUCTION:

In 2009, Ciric, Ignjatovi and Bagdanovi [1] introduced uniform fuzzy relations as a basis for defining such concept of a fuzzy function which would provide a correspondence between fuzzy functions and fuzzy equivalence relations, analogous to the correspondence between crisp functions and crisp equivalence relations. This was done, but also, it turned out that uniform fuzzy relations establish natural relationships between fuzzy partitions of two sets, some kind of "uniformity" between these fuzzy partitions. Roughly speaking, uniform fuzzy relations can be conceived as fuzzy equivalence relations which relate elements of two possibly different sets. In [1], uniform fuzzy relations were employed to solve some system of fuzzy relation equations, systems that have important applications in approximate reasoning, especially in fuzzy control. Afterwards, in [2], they were used to define and study fuzzy homomorphisms and fuzzy relational morphisms of algebras, and to esatablish relationships between fuzzy homomorphisms, fuzzy relational morphisms, and fuzzy congruences, analogous to relationships between homeomorphisms, relational morphisms, and congruences in classical algebra. In the same paper, fuzzy relational morphisms were also applied to deterministic fuzzy automata. It has been shown in [3] that fuzzy relational morphisms are the same as forward bisimulations (in the terminology used in this paper) when these two concepts are considered in the context of deterministic fuzzy automata. Further uniform fuzzy relations have shown their full strength in the study of equivalence between fuzzy automata, carried out in [3] (see also [4]).

SOME DEFINITIONS AND THEOREMS:

Let U and V be non-empty sets and let E and F fuzzy equivalences on U and V, respectively. If a fuzzy relation $R \in R(U, V)$ satisfies.

(EX1) $R(u_1, v) \otimes E(u_1, u_2) \leq R(u_2, v)$ for all $u_1, u_2 \in U$ and $v \in V$, then it is called extensional with respect to E, and if it satisfies

(EX2) $R(u, v_1) \otimes F(v_1, v_2) \leq R(u, v_2)$ for all $u \in U$ and $v_1, v_2 \in V$,

then it is called extensional with respect to F. If R is extensional with respect to E and F, and it satisfies

(PFF) $R(u, v_1) \otimes R(u, v_2) \leq F(v_1, v_2)$ for all $u \in U$ and $v_1, v_2 \in V$,

then it is called a partial fuzzy function with respect to E and F.

Partial fuzzy functions were introduced by Klawonn [5], and studied also by Demirci [6, 7]. By the adjunction property and symmetry, conditions (EX1) and (EX2) can be also written as

(EX1') $E(u_1, u_2) \leq (R(u_1, v) \leftrightarrow R(u_2, v))$ for all $u_1, u_2 \in U$ and $v \in V$;

(EX2') $F(v_1, v_2) \leq (R(u, v_1) \leftrightarrow R(u, v_2))$ for all $u \in U$ and $v_1, v_2 \in V$,

For any fuzzy relation $R \in R(U, V)$ we can define a fuzzy equivalence E_u^R on U by

$$E_u^R(u_1, u_2) = \bigwedge_{v \in V} (R(u_1, v) \leftrightarrow R(u_2, v)),$$

For all $u_1, u_2 \in U$, and a fuzzy equivalence E_V^R on V by

$$E_V^R(v_1, v_2) = \bigwedge_{u \in U} (R(u, v_1) \leftrightarrow R(u, v_2)),$$

For all $v_1, v_2 \in V$. They will be called fuzzy equivalences on U and V induced by R, and in particular, E_u^R will be called the kernel of R, and E_V^R the co-kernel of R. According to (EX1') and (EX2'), E_u^R and E_V^R are the greatest fuzzy equivalences on U and V, respectively, such that R is extensional with respect to them. Also, the fuzzy relation $R \circ R^{-1} \in R(U)$, will be called the projection of R on U, and $R^{-1} \circ R \in R(V)$, the projection of R on V.

A fuzzy relation $R \in R(U, V)$, is called just a partial fuzzy function if it is partial fuzzy function with respect to E_u^R and E_V^R [1]. Partial fuzzy functions were characterized in [1, 3] as follows:

Theorem 1.: Let U and V be non-empty sets and let $R \in R(U, V)$, be a fuzzy relation. Then the following conditions are equivalent:

(i) R is a partial fuzzy function;

(ii) R^{-1} is a partial fuzzy function;

(iii) $R^{-1} \circ R \leq E_V^R$;

(iv) $R \circ R^{-1} \leq E_u^R$;

(v) $R \circ R^{-1} \leq R \leq R$.

The name partial fuzzy function was introduced in [5], but it should be noted that the notion of a partial fuzzy function can not be considered as a natural analog of a partial function, because for a partial function relation R, its reverse R^{-1} is not necessarily a partial function. For the crisp counterpart of partial fuzzy functions has been called in [4] a partial uniform relation.

A fuzzy relation R ∈ R(U, V) is called an L-function if for any u ∈ U there exists v ∈ V such that R(u, v) = 1 [8], and it is called surjectiveif for any v ∈ V there exists u ∈ U such that R(u, v) =1, i.e., if R is an L-function. For a surjective fuzzy relation R ∈ R(U, V)we also say that it is a fuzzy relation of U onto V. If R is both an L-function and surjective, i.e., if both R and R^{-1} are L-functions, then R is called a surjective L-function. If for any u ∈ U there exists a unique v ∈ V such that R(u, v) =1, then R is called an F-function [9].

Let us note that a fuzzy relation R ∈ R(U, V) is an L-function if and only if there is a function ψ : U → V such that R(u, ψ (u)) = 1, for all u ∈ U (cf.[7, 8]). A function ψ with this property is called a crisp description of R, and we denote by CR(R) the set of all such functions.

An L-function which is a partial fuzzy function with respect to E and F is called a perfect fuzzy function with respect to E and F. Perfect fuzzy functions were introduced and stuided by Demiric [6. 7]. A fuzzy relations R ∈ R(U, V) which is a perfect fuzzy function with respect to E_u^R and E_V^R will be called just a perfect fuzzy function.

Let U and V be non-empty sets and let E be a fuzzy equivalence on V. An ordinary function ψ : U → V is called E-surjective if for any v ∈ V there exists u ∈ U such that E(ψ(u), v) =1. In other words, ψ is E-surjective if and only if ψ o $E^{\#}$ is an ordinary surjective function of U onto V/E, where $E^{\#}$: V → V/E is a function given by $E^{\#}$ (v) = E_v, for each v ∈ V. It is clear that ψ is an E-surjective function if and only if its image Im ψ has a non-empty intersection with every equivalence class of the crisp equivalence ker(E).

Let U and V be non-empty sets and let R ∈ R(U, V) be a partial fuzzy function. If, in addition, R is a surjective L-function, then it will be called a uniform fuzzy relation [1]. In other words, a uniform fuzzy relation is a perfect fuzzy function having the additional property that it is surjective. A uniform fuzzy relation that is also a crisp relation is called a uniform relation. The following characterizations of uniform fuzzy relations provided in [13] will be used in the further text.

Theorem 2.: Let u and V be non-empty sets and let R ∈ R(U, V) be a fuzzy relation. Then the following conditions are equivalent:

(i) R is a uniform fuzzy relation;

(ii) R^{-1} is a uniform fuzzy relation;

(iii) R is a surjective L-function and

$$R \circ R^{-1} \circ R = R; \qquad (1)$$

(iv) R is a surjective L-function and

$$E_u^R = R \circ R^{-1}; \quad (2)$$

(v) R is a surjective L-function and

$$E_V^R = R^{-1} \circ R; \quad (3)$$

(vi) R is an L-function, and for all $\psi \in CR(R)$, $u \in U$ and $v \in V$ we have that ψ is E_V^R surjective and

$$R(u, v) = E_V^R(\psi(u), v); \quad (4)$$

(vii) R is an L-function, and for all $\psi \in CR(R)$ and u_1 $u_2 \in U$ and we have that ψ is E_V^R surjective and

$$R(u_1, \psi(u_2)) = E_u^R(u_1, u_2). \quad (5)$$

Corollary 3.: [1] Let U and V be non-empty sets, and let $\varphi \in F(U \times V)$ be a uniform fuzzy relation. Then for all $\varphi \in CR(\varphi)$ and $u_1, u_2 \in A$ we have that

$$E_u^{\varphi}(u_1, u_2) = E_V^{\varphi}(\psi(u_1), \psi(u_2)). \quad (6)$$

A fuzzy relation $R \in R(U, V)$ is called an uniform FL-function if it is both a uniform fuzzy relation and an F-function, i.e., if it is a uniform fuzzy relation and E_V^R is a fuzzy equality (cf. [1]).

Let U and V be non-empty sets. Then a fuzzy relation $R \in R(U, V)$ is a uniform fuzzy relation if and only if its inverse relation R^{-1} is a uniform fuzzy relation. Moreover, by (iv) and (v) of Theorem 2, we have that the kernel of R^{-1} is the co-kernel of R, and conversely, the co-kernel of R^{-1} is the kernel of R, that is

$$E_V^{R^{-1}} = E_V^R \text{ and } E_u^{R^{-1}} = E_u^R. \quad (7)$$

The next theorem proved in [1, 3] will be very useful in our further work.

Theorem 4.: Let U and V be non-empty sets, let $R \in R(U, V)$ be a uniform fuzzy relation, let $E = E_u^R$ and $F = E_V^R$, and let a function $\tilde{R} : U/E \to V/F$ be defined by

$$\tilde{R}(E_u) = F_{\psi(u)}, \text{ for any } u \in U \text{ and } \psi \in CR(R). \quad (8)$$

Then $\tilde{R}$ is a well-defined function (it does not depend on the choice of $\psi \in CR(R)$ and $u \in U$), it is a bijective function of U/E onto V/F, and $(\tilde{R})^{-1} = \widetilde{R^{-1}}$.

The bijective function $\tilde{R}$ establishes some kind of "uniformity" between fuzzy partitions on U and V which correspond to fuzzy equivalences E_u^R and E_V^R, and for that reason these fuzzy relations are called uniform.

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