# Application of Elzaki Transform for Solving Population Growth and Decay Problems

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***Abstract: The population growth and decay problems arise in the field of chemistry, physics, biology, social science, zoology etc. In this paper, we used Elzaki transform for solving population growth and decay problems and some applications are given in order to demonstrate the effectiveness of Elzaki transform for solving population growth and decay problems.***

***Keywords: Elzaki transform, Inverse Elzaki transform, Population growth problem, Decay problem, Half-life.***

**I.Introduction:** The population growth (growth of a plant, or a cell, or an organ, or a species) is governed by the first order linear ordinary differential equation [1-10]

$$\frac{dN}{dt} = KN \ldots\ldots\ldots\ldots\ldots(1)$$

with initial condition as

$$N(t_0) = N_0 \ldots\ldots\ldots..(2)$$

where $K$ is a positive real number, $N$ is the amount of population at time $t$ and $N_0$ is the initial population at time $t_0$.

Equation (1) is known as the Malthusian law of population growth.

Mathematically the decay problem of the substance is defined by the first order linear ordinary differential equation [7, 9-10]

$$\frac{dN}{dt} = -KN \ldots\ldots\ldots\ldots\ldots(3)$$

with initial condition as

$$N(t_0) = N_0 \ldots\ldots\ldots..(4)$$

where N is the amount of substance at time $t, K$ is a positive real number and $N_0$ is the initial amount of the substance at time $t_0$.

In equation (3), the negative sign in the R.H.S. is taken because the mass of the substance is decreasing with time and so the derivative $\frac{dN}{dt}$ must be negative.

The Elzaki transform of the function $F(t)$ is defined as [11]:

$$E\{F(t)\} = v\int_0^\infty F(t)e^{-\frac{t}{v}}dt = T(v), \quad \text{where} \quad t \geq 0, 0 < k_1 \leq v \leq k_2 \ldots\ldots(6)$$

where $E$ is Elzaki transform operator.

The Elzaki transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Elzaki transform of the function $F(t)$.

Elzaki et al. [12] defined fundamental properties of Elzaki transform together with applications. HwaJoon Kim [13] gave the time shifting theorem and convolution for Elzaki transform. Elzaki and Ezaki [14] discussed the connections between Laplace & Elzaki transforms. Elzaki and Ezaki [15] used Elzaki transform for solving ordinary differential equation with variable coefficients. The solution of partial differential equations using Elzaki transform was given by Elzaki and Ezaki [16]. Shendkar and Jadhav [17] used Elzaki transform for the solution of differential equations. Aggarwal [18] discussed Elzaki transform of Bessel's functions.

The aim of this work is to finding the solution of population growth and decay problems using Elzaki transform without large computational work.

**II. Linearity Property of Elzaki Transform [18]:**

If $E\{F(t)\} = H(v)$ and $E\{G(t)\} = I(v)$ then

$$E\{aF(t) + bG(t)\} = aE\{F(t)\} + bE\{G(t)\}$$

$$\Rightarrow E\{aF(t) + bG(t)\} = aH(v) + bI(v),$$

where $a, b$ are arbitrary constants.

**III. Elzaki Transform of Some Elementary Functions [11-1 2, 18]:**

| S.N. | $F(t)$ | $E\{F(t)\} = T(v)$ |
|---|---|---|
| 1. | $1$ | $v^2$ |
| 2. | $t$ | $v^3$ |
| 3. | $t^2$ | $2!\,v^4$ |
| 4. | $t^n, n \in N$ | $n!\,v^{n+2}$ |
| 5. | $t^n, n > -1$ | $\Gamma(n+1)v^{n+2}$ |
| 6. | $e^{at}$ | $\frac{v^2}{1-av}$ |
| 7. | $sinat$ | $\frac{av^3}{1+a^2v^2}$ |
| 8. | $cosat$ | $\frac{v^2}{1+a^2v^2}$ |
| 9. | $sinhat$ | $\frac{av^3}{1-a^2v^2}$ |
| 10. | $coshat$ | $\frac{v^2}{1-a^2v^2}$ |

**IV. Inverse Elzaki Transform [18]:**

If $E\{F(t)\} = T(v)$ then $F(t)$ is called the inverse Elzaki transform of $T(v)$ and mathematically it is defined as

$$F(t) = E^{-1}\{T(v)\}$$

where $E^{-1}$ is the inverse Elzaki transform operator.

**V. Inverse Elzaki Transform of Some Elementary Functions [18]:**

| S.N. | $T(v)$ | $F(t)=E^{-1}\{T(v)\}$ |
|---|---|---|
| 1. | $v^2$ | $1$ |
| 2. | $v^3$ | $t$ |
| 3. | $v^4$ | $\frac{t^2}{2!}$ |
| 4. | $v^{n+2}, n\in N$ | $\frac{t^n}{n!}$ |
| 5. | $v^{n+2}, n>-1$ | $\frac{t^n}{\Gamma(n+1)}$ |
| 6. | $\frac{v^2}{1-av}$ | $e^{at}$ |
| 7. | $\frac{v^3}{1+a^2v^2}$ | $\frac{sinat}{a}$ |
| 8. | $\frac{v^2}{1+a^2v^2}$ | $cosat$ |
| 9. | $\frac{v^3}{1-a^2v^2}$ | $\frac{sinhat}{a}$ |
| 10. | $\frac{v^2}{1-a^2v^2}$ | $coshat$ |

**VI. Elzaki Transform of The Derivatives of The Function $F(t)$ [13, 17-18]:**

If $E\{F(t)\}=T(v)$ then

a) $E\{F'(t)\}=\frac{T(v)}{v}-vF(0)$

b) $E\{F''(t)\}=\frac{T(v)}{v^2}-vF'(0)-F(0)$

**VII. Elzaki Transform for Population Growth Problem:**

In this section, we present Elzaki transform for population growth problem given by (1) and (2).

Applying the Elzaki transform on both sides of (1), we have

$$E\left\{\frac{dN}{dt}\right\}=KE\{N(t)\}\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

Now applying the property, Elzaki transform of derivative of function, on (5), we have

$$\frac{1}{v}E\{N(t)\}-vN(0)=KE\{N(t)\}\ldots\ldots(6)$$

Using (2) in (6) and on simplification, we have

$$\left(\frac{1}{v}-K\right)E\{N(t)\}=vN_0$$

$$\Rightarrow E\{N(t)\}=\frac{v^2N_0}{(1-Kv)}\ldots\ldots\ldots\ldots\ldots(7)$$

Operating inverse Elzaki transform on both sides of (7), we have

$$N(t)=E^{-1}\left\{\frac{N_0v^2}{(1-Kv)}\right\}$$

$$\Rightarrow N(t)=N_0E^{-1}\left\{\frac{v^2}{(1-Kv)}\right\}$$

$$\Rightarrow N(t)=N_0e^{Kt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

which is the required amount of the population at time $t$.

**VIII. Elzaki Transform for Decay Problem:**

In this section, we present Elzaki transform for decay problem which is mathematically given by (3) and (4).

Applying the Elzaki transform on both sides of (3), we have

$$E\left\{\frac{dN}{dt}\right\}=-KE\{N(t)\}\ldots\ldots\ldots\ldots\ldots\ldots.(9)$$

Now applying the property, Elzaki transform of derivative of function, on (9), we have

$$\frac{1}{v}E\{N(t)\}-vN(0)=-KE\{N(t)\}\ldots(10)$$

Using (4) in (10) and on simplification, we have

$$\left(\frac{1}{v}+K\right)E\{N(t)\}=vN_0$$

$$\Rightarrow E\{N(t)\}=\frac{v^2N_0}{(1+Kv)}\ldots\ldots\ldots\ldots\ldots(11)$$

Operating inverse Elzaki transform on both sides of (11), we have

$$N(t)=E^{-1}\left\{\frac{N_0v^2}{(1+Kv)}\right\}$$

$$\Rightarrow N(t)=N_0E^{-1}\left\{\frac{v^2}{(1+Kv)}\right\}$$

$$\Rightarrow N(t)=N_0e^{-Kt}\ldots\ldots\ldots\ldots\ldots(12)$$

which is the required amount of substance at time $t$.

**IX. Applications:**

In this section, some applications are given in order to demonstrate the effectiveness of Elzaki transform for solving population growth and decay problems.

**Application:1** The population of a city grows at a rate proportional to the number of people presently living in the city. If after two years, the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the city.

This problem can be written in mathematical form as:

$$\frac{dN(t)}{dt}=KN(t)\ldots\ldots\ldots\ldots\ldots.(13)$$

where $N$ denote the number of people living in the city at any time t and $K$ is the constant of proportionality. Consider $N_0$ is the number of people initially living in the city at $t=0$.

Applying the Elzaki transform on both sides of (13), we have

$$E\left\{\frac{dN}{dt}\right\}=KE\{N(t)\}\ldots\ldots\ldots\ldots\ldots\ldots.(14)$$

Now applying the property, Elzaki transform of derivative of function, on (14), we have

$$\frac{1}{v}E\{N(t)\}-vN(0)=KE\{N(t)\}\ldots\ldots(15)$$

Since at $t=0, N=N_0$, so using this in (15), we have

$$\left(\frac{1}{v}-K\right)E\{N(t)\}=vN_0$$

$$\Rightarrow E\{N(t)\}=\frac{v^2N_0}{(1-Kv)}\ldots\ldots\ldots\ldots..(16)$$

Operating inverse Elzaki transform on both sides of (16), we have

$$N(t)=E^{-1}\left\{\frac{N_0v^2}{(1-Kv)}\right\}$$

$$\Rightarrow N(t)=N_0E^{-1}\left\{\frac{v^2}{(1-Kv)}\right\}$$

$$\Rightarrow N(t)=N_0e^{Kt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Now at $=2$, $N=2N_0$, so using this in (17), we have

$$2N_0=N_0e^{2K}$$

$$\Rightarrow e^{2K}=2$$

$$\Rightarrow K=\frac{1}{2}log_e2=0.347\ldots\ldots\ldots\ldots\ldots(18)$$

Now using the condition at $=3$, $N=20{,}000$, in (17), we have

$$20{,}000=N_0e^{3K}\ldots\ldots\ldots\ldots.(19)$$

Putting the value of $K$ from (18) in (19), we have

$$20{,}000=N_0e^{3\times0.347}$$

$$\Rightarrow 20{,}000=2.832N_0$$

$$\Rightarrow N_0\simeq 7062\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

which are the required number of people initially living in the city.

**Application:2** A radioactive substance is known to decay at a rate proportional to the amount present. If initially there is 100 milligrams of the radioactive substance present and after two hours it is observed that the radioactive substance has lost 10 percent of its original mass, find the half life of the radioactive substance.
This problem can be written in mathematical form as:

$$\frac{dN(t)}{dt} = -KN(t) \ldots\ldots\ldots\ldots\ldots.(21)$$

where $N$ denote the amount of radioactive substance at time $t$ and $K$ is the constant of proportionality. Consider $N_0$ is the initial amount of the radioactive substance at time $t = 0$.
Applying the Elzaki transform on both sides of (21), we have

$$E\left\{\frac{dN}{dt}\right\} = -KE\{N(t)\} \ldots\ldots\ldots\ldots\ldots\ldots.(22)$$

Now applying the property, Elzaki transform of derivative of function, on (22), we have

$$\frac{1}{v}E\{N(t)\} - vN(0) = -KE\{N(t)\} \ldots(23)$$

Since at $t = 0, N = N_0 = 100$, so using this in (23), we have

$$\frac{1}{v}E\{N(t)\} - 100v = -KE\{N(t)\}$$

$$\Rightarrow \left(\frac{1}{v} + K\right) E\{N(t)\} = 100v$$

$$\Rightarrow\ E\{N(t)\} = \frac{100v^2}{(1 + Kv)} \ldots\ldots\ldots\ldots..(24)$$

Operating inverse Elzaki transform on both sides of (24), we have

$$N(t) = E^{-1}\left\{\frac{100v^2}{(1 + Kv)}\right\}$$

$$= 100E^{-1}\left\{\frac{v^2}{(1 + Kv)}\right\}$$

$$\Rightarrow N(t) = 100e^{-Kt} \ldots\ldots..(25)$$

Now at $t = 2$, the radioactive substance has lost 10 percent of its original mass 100 mg so $N = 100 - 10 = 90$, using this in (25), we have

$$90 = 100e^{-2K}$$

$$\Rightarrow e^{-2K} = 0.90$$

$$\Rightarrow K = -\frac{1}{2}log_e 0.90 = 0.05268 \ldots\ldots\ldots(26)$$

We required $t$ when $N = \frac{N_0}{2} = \frac{100}{2} = 50$ so from (25), we have

$$50 = 100e^{-Kt} \ldots\ldots\ldots\ldots\ldots.(27)$$

Putting the value of $K$ from (26) in (27), we have

$$50 = 100e^{-0.05268t}$$

$$\Rightarrow e^{-0.05268t} = 0.50$$

$$\Rightarrow t = -\frac{1}{0.05268}log_e 0.50$$

$$\Rightarrow t = 13.157 hours \ldots\ldots\ldots\ldots..(28)$$

which is the required half-time of the radioactive substance.

**X. Conclusion:** In this paper, we have successfully developed the Elzaki transform for solving the population growth and decay problems. The given applications show that the effectiveness of Elzaki transform for solving population growth and decay problems. The proposed scheme can be applied for the continuous compound interest and heat conduction problems.

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