Some Properties of Kenmotsu Manifold Admitting aSemi-symmetric Non-metric Connection

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Abstract

The object of the present paper is to study A study of Some Properties of Kenmotsu Manifold Admitting a Semi-symmetric Non-metric Connection and some properties of quarter-symmetric non-metric connection.

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1 Introduction

Let (M_n, g) be a Riemannian manifold of dimension n. A linear connection ∇ in (M_n, g), whose torsion tensor T of type $(1, 2)$ is defined as

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (1.1)$$

for arbitrary vector fields X and Y, is said to be torsion free or symmetric if T vanishes, otherwise it is non-symmetric. If the connection ∇ satisfy $\nabla g = 0$ in (M_n, g), then it is called metric connection, otherwise it is non-metric.

In 1972, K. Kenmotsu [18] introduced a class of contact Riemann manifold known as Kenmotsu Manifold. He studied that if a Kenmotsu manifold satisfy the condition $R(X, Y)Z = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type (1, 3) and $R(X, Y)Z$ is derivative of tensor algebra at each point of the tangent space. Several properties of Kenmotsu Manifold have been studied by De and Pathak [7], De [8], Sinha and Srivastav, De and Pathak [7]and many others. Ozgur studied generalised recurrent Kenmotsu manifold and proved that if M be a generalised recurrent Kenmotsu manifold and generalised Ricci Recurrent Kenmotsu manifold then $\beta = \alpha$ holds on M . Sular studies the generalised recurrent and generalised Ricci recurrent Kenmotsu manifol with respect to semi symmetric metric connection and proved that $\beta = 2\alpha$ where α and β are smooth functions and M is generalised recurrent and generalised Ricci recurrent Kenmotsu manifold admitting a semi-symmetric connection. In the present chapter, we studied the properties of semi-symmetric non-metric connection in Kenmotsu manifold.

2 Preliminaries

An n-dimensional Riemannian manifold (M_n, g) of class C^∞ with a 1-form η, the associated vector field ξ and a $(1, 1)$ tensor field φ satisfying

$$\varphi^2 X + X = \eta(X)\xi, \quad (2.1)$$

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1, \quad (2.2)$$

for arbitrary vector field X, is called an almost contact manifold. This system (φ, ξ, η) is called an almost contact structure to M_n [?]. If the associated Riemannian metric g in M_n satisfy

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X, Y in M_n, then (M_n, g) is said to be an almost contact metric manifold. Putting ξ for X in (2.3) and using (2.2), we obtain

$$g(\xi, Y) = \eta(Y). \tag{2.4}$$

Also,

$$\phi(X, Y) \overset{\text{def}}{=} g(\varphi X, Y) \tag{2.5}$$

gives

$$\phi(X, Y) + \phi(Y, X) = 0. \tag{2.6}$$

where $\phi = d\eta$ is 2-form.

If moreover

$$(D_X\varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.7}$$

$$D_X\xi = X - \eta(X)\xi, \tag{2.8}$$

hold in (M_n, g), where D being the Levi-Civita connection of the Riemannian metric g, then (M_n, g) is called a Kenmotsu manifold [18]. Also the following relations hold in a *Kenmotsu manifold*

$$K(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$K(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.11}$$

$$(D_X\eta)(Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.12}$$

for arbitrary vector fields X and Y, where K and S denote the Riemannian curvature and Ricci tensors of the connection D respectively.

Theorem 2.1. Let M_n be an n-dimensional Kenmotsu manifold equipped with a semi- symmetric non-metric connection ∇, then the scalar curvature with respect to semi- symmetric non-metric connection is equal to scalar curvature with respect to Levi-Civita connection.

Definition 2.2. A non-flat n-dimensional differentiable manifold M_n, $(n > 3)$, is called pseudo symmetric if there is a 1-form A on M_n such that

$$\begin{aligned}(D_X K)(Y, Z)W &= 2A(X)K(Y, Z)W + A(Y)K(X, Z)W + A(Z)K(Y, X)W \\ &\quad + A(W)K(Y, Z)X + g(K(Y, Z)W, X)\rho_1, \end{aligned} \tag{2.13}$$

where D is the Levi-Civita connection and X, Y, Z and W are arbitrary vector fields on M_n. The vector field ρ_1 associated with the 1-form A is defined by $A(X) = g(X, \rho_1)$.

Definition 2.3. A non-flat n-dimensional differentiable manifold M_n, $(n > 3)$, is calledweakly symmetric if there are 1-forms A, B, C and D on M_n such that

$$\begin{aligned}(D_X K)(Y, Z)W &= A(X)K(Y, Z)W + B(Y)K(X, Z)W + C(Z)K(Y, X)W \\ &\quad + D(W)K(Y, Z)X + g(K(Y, Z)W, X)\sigma, \end{aligned} \tag{2.14}$$

where X, Y, Z and W are arbitrary vector fields on M_n. The vector field σ associated with the 1-form p is defined as $p(X) = g(X, \sigma)$. A weakly symmetric manifold M_n is said to be pseudo symmetric if $B = C = D = A$, $\sigma = \rho_1$ and A is replaced by $2A$, locally symmetric if $A = B = C = D = 0$ and $\sigma = 0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms A, B, C and D is not zero or $\sigma /= 0$.

Definition 2.4. A non flat n-dimensional differentiable manifold M_n, $(n > 3)$, is called weakly Ricci-symmetric if there are 1-forms α, β and γ on M_n such that

$$(D_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \gamma(Z)S(X, Y), \tag{2.15}$$

where X, Y and Z are arbitrary vector fields on M_n. A weakly Ricci-symmetric manifold M_n is called pseudo Ricci-symmetric if $\alpha = \beta = \gamma$.

Similarly we define the following definitions:

Definition 2.5. A non-flat n-dimensional differentiable manifold M_n, $(n > 3)$, is called weakly symmetric with respect to the semi-symmetric non-metric connection ∇ if there are 1-forms A, B, C and D on M_n such that

$$\begin{aligned}(\nabla_X R)(Y, Z)W &= A(X)R(Y, Z)W + B(Y)R(X, Z)W + C(Z)R(Y, X)W \\ &\quad + D(W)R(Y, Z)X + g(R(Y, Z)W, X)\sigma, \end{aligned} \tag{2.16}$$

where X, Y, Z and W are arbitrary vector fields on M_n and the 1-forms A, B, C, D andthe vector field σ are defined previously.

Definition 2.6. A non-flat n-dimensional differentiable manifold M_n, $(n > 3)$, is called weakly Ricci-symmetric with respect to the semi-symmetric non-metric connection ∇ if there are 1-forms α, β and γ on M_n such that

$$(\nabla_X \tilde{S})(Y, \tilde{Z}) = \alpha(X)\tilde{S}(Y, Z) + \beta(Y)\tilde{S}(X, Z) + \gamma(Z)\tilde{S}(X, Y), \tag{2.17}$$

where X, Y and Z are arbitrary vector fields on M_n.

Contracting (2.16) with Y, we get

$$\begin{aligned}(\nabla_X\tilde{S})(Z,W) = \; & A(X)\tilde{S}(Z,W) + B(R(X,Z)W) + C(Z)\tilde{S}(W,X) \\ & + D(W)\tilde{S}(X,Z) + p(R(X,W)Z), \end{aligned} \tag{2.18}$$

where p is defined as $p(X) = g(X,\sigma)$ for arbitrary vector field X.

Özgür [25] considered weakly symmetric and weakly Ricci-symmetric Kenmotsu manifold and proved the following theorems:

Theorem 2.7. There is no weakly symmetric Kenmotsu manifold M, $(n > 3)$, unless $A + C + D$ is everywhere zero.

Theorem 2.8. There is no weakly Ricci-symmetric Kenmotsu manifold M, $(n > 3)$, unless $\alpha + \beta + \gamma$ is everywhere zero.

Sular [30] considered weakly symmetric and weakly Ricci-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection and proved the following results:

Theorem 2.9. There is no weakly symmetric Kenmotsu manifold M admitting a semi- symmetric metric connection, $(n > 3)$, unless $A + C + D$ is everywhere zero.

Theorem 2.10. There is no weakly Ricci-symmetric Kenmotsu manifold M admitting a semi-symmetric metric connection, $(n > 3)$, unless $\alpha + \beta + \gamma$ is everywhere zero.

Now we consider the weakly symmetric and weakly Ricci-symmetric Kenmotsu manifold admitting the semi-symmetric non-metric connection ∇ and prove the following theorem:

Theorem 2.11. Let M_n, $(n > 3)$ be an n-dimensional weakly symmetric Kenmotsu manifold admitting a semi-symmetric non-metric connection ∇ then there is no M_n, unless $A + C + D$ is everywhere zero.

Theorem 2.12. Let M_n, $(n > 3)$, be an n-dimensional weakly Ricci-symmetric Kenmotsu manifold admitting a semi-symmetric non-metric connection then there is no M_n, unless $\alpha + \beta + \gamma$ is everywhere zero where α, β and γ are parameters of the manifold.

Proof. Putting $Z = \xi$ in (2.17) and then using (2.11), (**??**) and (**??**), we find

$$\begin{aligned}(\nabla_X\tilde{S})(Y,\xi) = \; & -(n-1)\alpha(X)\eta(Y) - (n-1)\beta(Y)\eta(X) \\ & +\gamma(\xi)S(X,Y) + (n-1)\gamma(\xi)g(\varphi X,Y). \end{aligned} \tag{2.19}$$

In consequence of (2.19), we have

$$\begin{aligned} & -(n-1)g(X,Y) + 2(n-1)g(\varphi X,Y) - S(X,Y) \\ & = -(n-1)\alpha(X)\eta(Y) - (n-1)\beta(Y)\eta(X) \\ & +\gamma(\xi)S(X,Y) + (n-1)\gamma(\xi)g(\varphi X,Y). \end{aligned} \tag{2.20}$$

Taking $X = Y = \xi$ in (2.20) and using (2.2), (2.4) and (2.11), we find

$$\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0. \tag{2.21}$$

Replacing X by ξ in (2.20) and using (2.2), (2.4), (2.11) and (2.21), we get

$$\beta(Y) = \beta(\xi)\eta(Y). \tag{2.22}$$

Again replacing Y by ξ in (2.20) and using (2.2), (2.4), (2.11) and (2.21), we obtain

$$\alpha(X) = \alpha(\xi)\eta(X). \tag{2.23}$$

From (2.2), (2.8), (2.11), (2.12)

$$(\nabla_\xi \tilde{S})(\xi, X) = 0. \tag{2.24}$$

In view of (2.17), above equation becomes

$$\alpha(\xi)\tilde{S}(\xi, X) + \beta(\xi)\tilde{S}(\xi, X) + \gamma(X)\tilde{S}(\xi, \xi) = 0. \tag{2.25}$$

With the help of (2.2), (2.4), (2.11) and gives

$$\gamma(X) = \gamma(\xi)\eta(X). \tag{2.26}$$

Adding (2.22), (2.23) and (2.26) and using (2.21), the theorem follows. □

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