Posets and Forbidden induced subgraph of the Line graph

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Abstract : *The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. As a continuation of the study of 2-colored and 3-colored diagrams we characterize some forbidden ⊲ - preserving subposets of the posets whose cover-incomparability graph contains one of the forbidden induced subgraph of the line graph.*

IndexTerms -. Cover-incomparabilitygraph, Blockgraph, Line graph, Poset

INTRODUCTION

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10].Let P = (V;_≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ϵ V we say that v *covers* u in P if u < v and there is no w in V with u <w < v. If u ≤ v we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write u ⊲ v if v covers u; and u ⊲⊲ v if u is below v but not covered by v. By u ‖ v we denote that u and v are incomparable. Let V' be a nonempty subset of V. Then there is a natural poset Q =(V';≤ ′), where u ≤ ′ v if and only if u ≤ v for any u,v $\epsilon V'$. The poset Q is called a *subposet* of P and its notation is simplified to Q = (V';≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if u ⊲ v, v ⊲ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2**]** Let P be a poset and G_P its C-I graph. Then
(i) G_P is connected;
(ii) vertices in an independent set of G_P lie on a common chain of P;
(iii) an antichain of P corresponds to a complete subgraph in G_P ;
(iv) G_P contains no induced cycles of length greater than 4.

II. 3-colored diagrams

A 3-coloured diagram Q in [13] is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H. Let u,v,w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the ⊲ - preserving subposets. It is possible to have u ⊲ v, u ‖ v, v ⊲ w and v ‖ w. Each case will appear as a ⊲ - preserving subposet of four different posets. If u ⊲ v and v ⊲ w in a subposet, then u ⊲ v ⊲ w is a chain in the subposet and u,v,w is an induced path in H. If there is either u ‖ v or v ‖ w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v,w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u,v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u,w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams.

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathscr{P}$ = {P | P is a poset with G_{TP} ϵ G }. Then $\mathscr{P}$ has a characterization by forbidden ⊲ - preserving subposets.

Theorem 3: (Theorem 7.1.8, [7]) Let G be a graph. Then G is a line graph if and only if G contains none of the nine forbidden graphs of Figure 1 as an induced subgraph

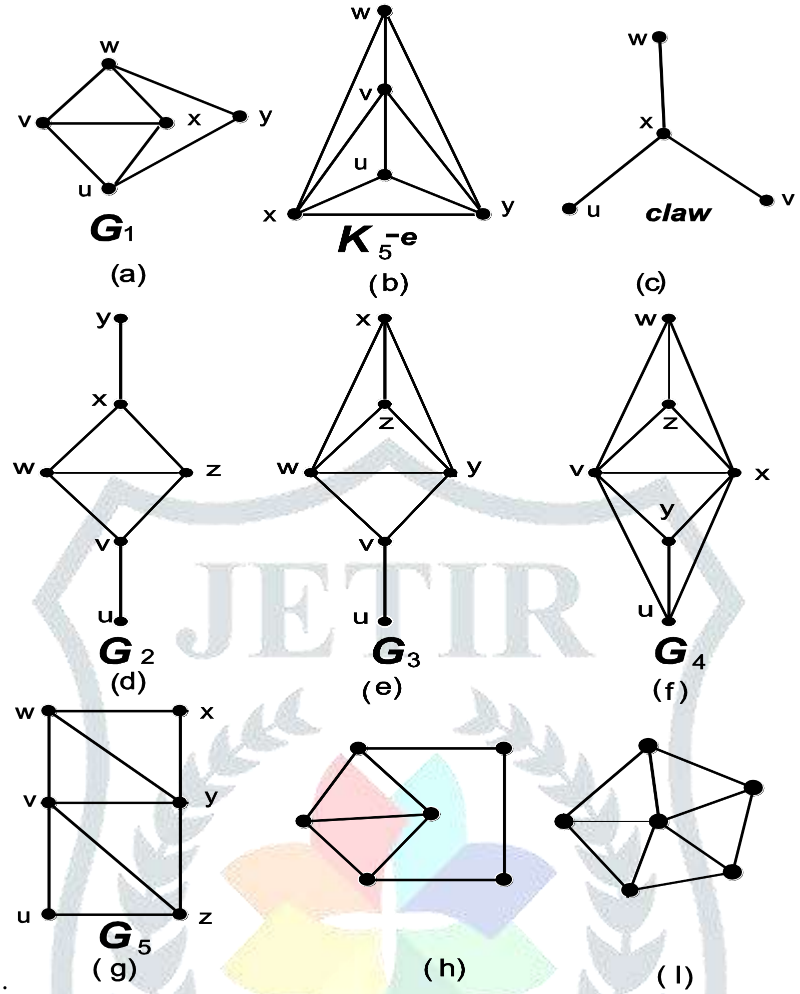

FIGURE 1: NINE FORBIDDEN INDUCED SUB GRAPHS OF LINE GRAPH

Theorem 4: (Theorem 4.1,[12]) Let P be a poset. Then G_P is cograph if and only if P contains none of $T_1, \ldots, T_7$, depicted in Figure 2,

and no duals of T_2 and T_5 as ◃- preserving subposet.

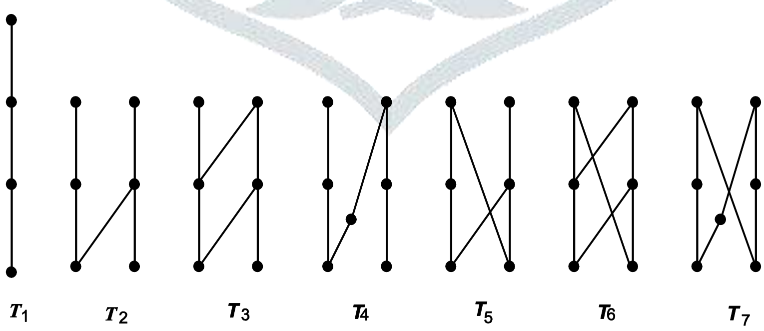

Figure 2: Forbidden ◃ - preserving subposets for C-I cographs

Theorem 5: (Theorem 4,[13]) If P is a poset, then G_P is cograph if and only if P does not contain T_1 from Figure 1 and no 3-colored diagram Q_C from Figure 3 and its dual are ◃ - preserving subposets .

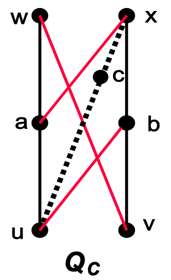

Figure 3: Forbidden $\lhd$ - preserving 3-colored subposets for C-I cographs

We consider some subposets to be forbidden so that its C-I graphs belong to the graph family $\mathcal{F}(G_5)$ of G_5 in Figure 1

III. $\lhd$ - preserving subposets of posets whose C-I graphs belong to the family $\mathcal{F}(G_5)$

We have the following theorem regarding the graph family $\mathcal{F}(G_5)$

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Theorem 6: If P is a poset, then G_P belongs to $\mathcal{F}$**G_5)** if and only if P contains the 3-colored diagrams Q_7 and Q_8 from Figure 4 and $\lhd$-preserving subposets U_1 , U_2 from Figure 5 and their duals.

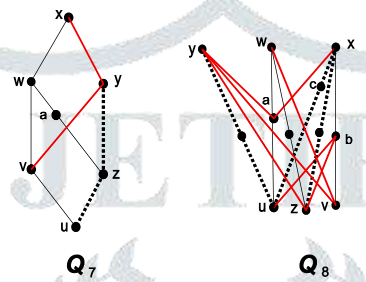


Figure 4:Forbidden 3-colored diagrams for posets whose C-I graphs contains G_5, depicted in Figure 1 (g).

Proof. Suppose P contains 3-colored diagrams Q_7, Q_8 from Figure 4 and $\lhd$-preserving subposets U_1, U_2 from Figure 5. Then, clearly G_P contains the graph from Figure 1(g) as an induced subgraph.

Conversely, suppose $G_P \epsilon$ $\mathcal{F}$**G_4).** Then G_P contains an induced subgraph isomorphic to G_5 as shown in Figure 1(g), with vertices labeled by u, v, w, x, y and z. There are four induced P_4 in G_P induced by vertex sets {u, v, w, x}, {u, v, y, x}, {z, v, w, x} and {u, z, y, x}. Without loss of generality, we consider the P_4 induced by the vertices u, v, w, x in G_5. Then by Theorem 5, there exists either a chain u $\lhd$ v $\lhd$ w $\lhd$ x in P or there exists the 3-colored diagram isomorphic to Q_C in P.

Case (1): The P_4 in G_5 induced by the vertices u, v, w and x is formed by the chain u $\lhd$ v $\lhd$ w $\lhd$ x in the poset P. Since z is adjacent to v in the graph G_5, either v and z are in a covering relation or these vertices are incomparable in P.

Subcase (1.1): v || z.

Since there is a path of length two from z to w in G_5, there must be a chain from z to w, let the chain be through the point a defined by normal edges in P. Consider the vertex y in G_5. There are two possibilities for y with respect to v. Either v $\lhd$ y or v || y (y $\not\lhd$ v, since w and y are adjacent in G_5).

Subcase (1.1.1): v $\lhd$ y and y $\lhd$ x.

Subcase (1.1.2): v $\lhd$ y and y || x.

In the posets described by the subcases (1.1.1), (1.1.2), corresponding to the adjacency relations among the vertices u, v, w, x, y and z in the graph G_5, satisfy the 3-colored poset Q_7 and we are done.

Subcase (1.1.3): v || y and y $\lhd$ x.

Subcase (1.1.4): v || y and y || x.

In subcases (1.1.3) and (1.1.4), there is no chain from u to y, but since there is a path of length two from u to y in G_5, there must be a chain from u to y, we allow a dashed line from u to y through z. y and x can have both possibilities, namely y $\lhd$ x or y || x and hence the edge xy can also be represented by red edge in the poset P. This situation is represented in the 3-colored diagram Q_7 shown in Figure 4 .

Subcase (1.2): z $\lhd$ v.

Since there is a path of length two from u to y in G_5, there must be a chain from u to y through the point b defined by normal edges in P. In this case, y and x can have both possibilities, namely y $\lhd$ x or y || x. This is represented by the subposets U_1 and U_2 shown in Figure .5

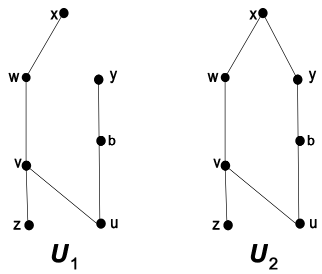
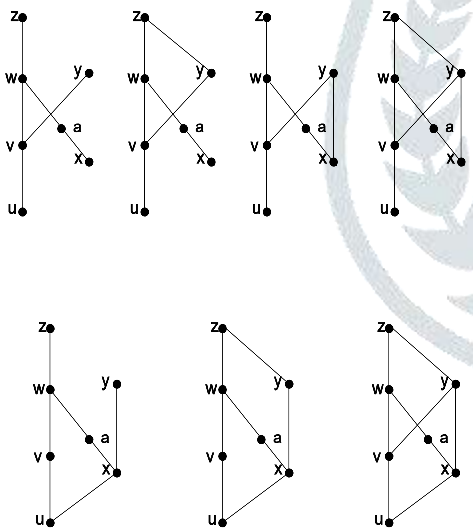


Figure 5:Forbidden subposets whose C-I graphs contains G_5, depicted in Figure 1(g).

Case (2): The P_4 in G_5 induced by the vertices u,v,w and x is formed by two chains of length 3 as in the poset P as shown in Figure 2.

By Theorem 5, we have that the set {u, v, w, x} will form the 3-colored diagram Q_C in Figure 3. Now we consider the vertices y and z in G_5 and find all the possibilities that these vertices can appear in the 3-colored diagram Q_C. Let the chain from z to w be defined by normal edges in P as described in Case(1). Since there is a path of length two from z to x and a path of length three from u to x in G_5, there must be chains of length three from z to x, and u to x in P. If both these chains pass through b in Q_C, then both the vertices are in a covering relation with b ($x \nprec z$ and $x \nprec u$, since z is adjacent to v and w is adjacent to x in G_5). Otherwise, there must be dashed lines between z and x, and u and x representing a chain of length 3 between z and x, and u and x respectively. Similar is the case between u and y in the graph G_5. Therefore, there must be a chain of length three from u to y in P. If the chain passes through a, then there is a covering relation between a and y ($y \nprec u$ since y and w are adjacent in G_5). Otherwise there must be a dashed line between u and y representing a chain of length 3 between u and y. Since vw, vy and zy are edges in G_5, there are three cases, either $v \lhd w$ or $v \parallel w$, $v \lhd y$ or $v \parallel y$ and $z \lhd y$ or $z \parallel y$ and hence these edges are red. From the above discussion, analyzing all the possibilities in which the vertices y and z can be related with the 3-colored diagram Q_C, it can be verified easily that we obtain the 3-colored diagram Q_8 in Figure 4, which is an extension of Q_C. Thus we have completed all the cases in which the vertices of the graph G_5 can appear in the poset P, which completes the proof of the theorem.



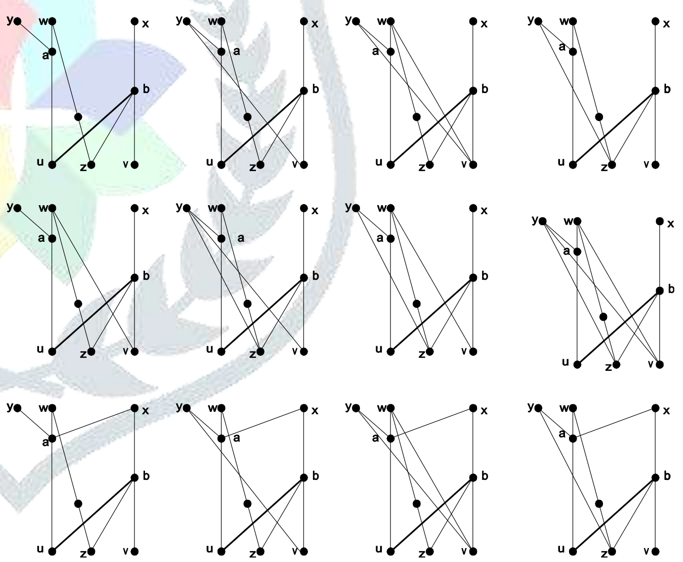
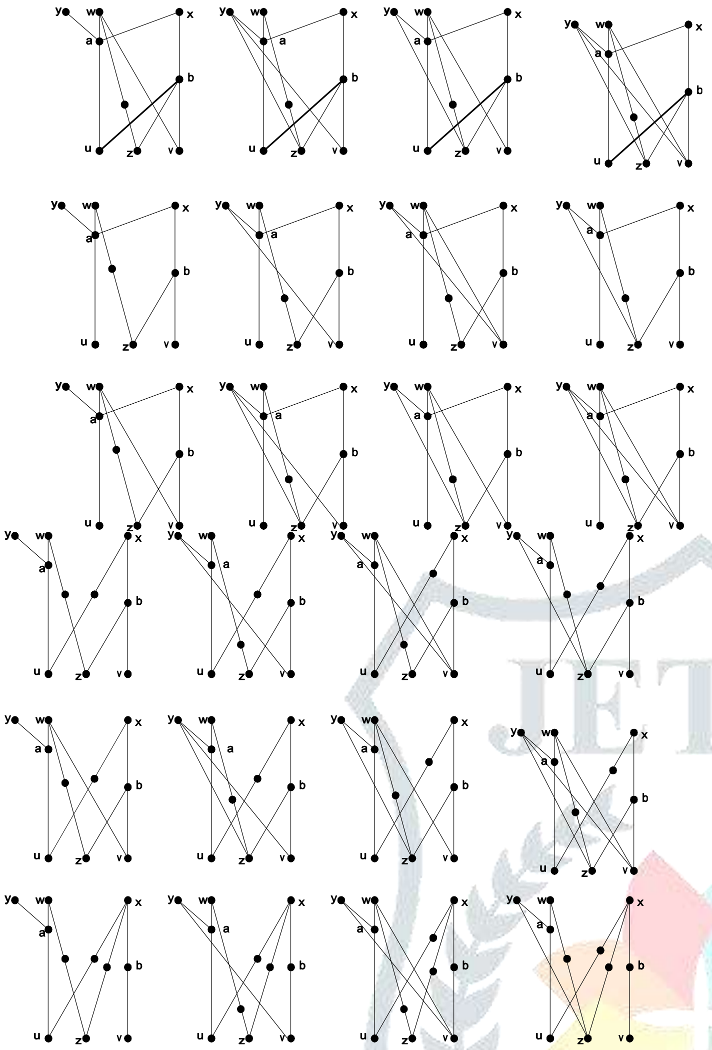


Figure 6:$\lhd$ - preserving subposets corresponding to **Q_7** $\lhd$ - preserving subposets corresponding to **Q_8**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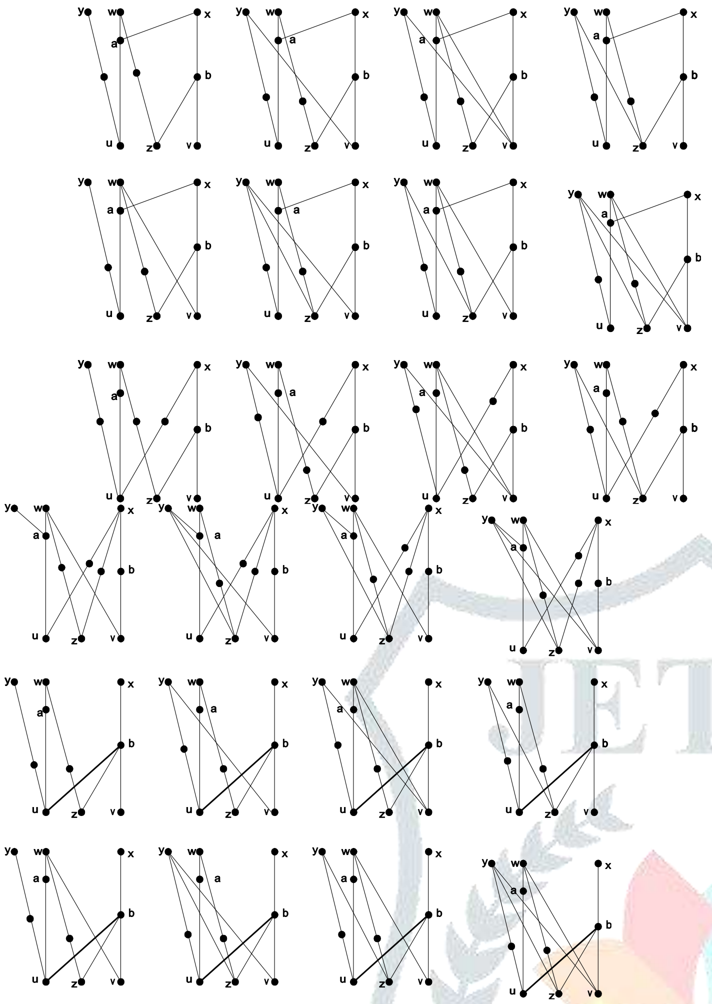

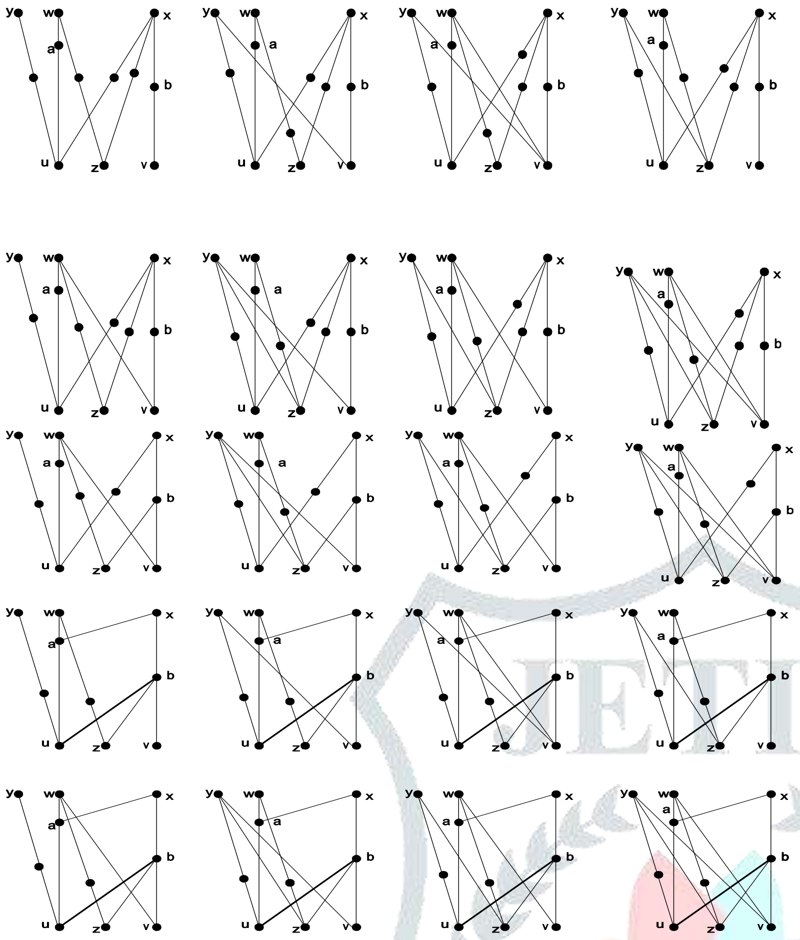

Figure 7:$\lhd$ - preserving subposets corresponding to $\mathbf{Q_8}$

Remarks

The number of forbidden $\lhd$ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. Here we characterize forbidden $\lhd$ - preserving subposets of G_5 in Figure1 and introduce the idea of 3-colored diagrams to minimize the list of subposets.

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