NON-DIFFERENTIABLE CONTINUOUS PROGRAMMING PROBLEMS

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Abstract : In this paper we consider a non-differentiable analogue of the variational problem treated by Mond and Hanson, and derive optimality criteria and consequently study duality for this problem.

Keywords : non-differentiable, non-differentiable continuous programming problems, optimality conditions, positive semidefinite.

INTRODUCTION

In this paper we consider a non-differentiable analogue of the variational problem treated by Mond and Hanson [1], and derive optimality criteria and consequently study duality for this problem. Here non-differentiability enters due to the square root of certain quadratic form which appears in the integrand of the objective functional. These duality results heavily depend on Fritz John type necessary optimality conditions which are established for a class of non-differentiable continuous programming problems. It is discussed that these results can be regarded as a dynamic generalization of Mond's results [2] for a class of non-differentiable mathematical programs.

NON-DIFFERENTIABLE CONTINUOUS PROGRAMMING PROBLEMS

Consider the following pair of non-differentiable continuous programming problems.

$$\text{Primal (P)}: \text{ Minimize } \emptyset_1(x) = \int_a^b [f(t, x(t), \dot{x}(t)) + (x(t)^T B(t) x(t)^{½}]d \qquad (1)$$

subject to,

$$x(a) = \alpha, x(b) = \beta \qquad (2)$$

$$g\big(t, x(t), \dot{x}(t)\big) \geq 0 \ (a \leq t \leq b) \ (3)$$

$$\text{Dual (D)}: \text{Maximize } \psi_1(x, \lambda, z) = \int_a^b [f(t, x(t), \dot{x}(t)) + x(t)^T B(t) z(t) - \lambda(t) g(t, x(t), \dot{x}(t))]dt \ (4)$$

Subject to,

$$x(a) = \alpha, x(b) = \beta \qquad (5)$$

$$f_x\big(t, x(t), \dot{x}(t)\big) + z(t)^T B(t) - \lambda(t) g_x\big(t, x(t), \dot{x}(t)\big) = D[f_{\dot{x}}(t, x(t), \dot{x}(t)) - \lambda(t) g_{\dot{x}}(t, x(t), \dot{x}(t)], (a \leq t \leq b) \qquad (6)$$

$$z(t)^T B(t) z(t) \leq 1, (a \leq t \leq b) \ (7)$$

$$\lambda(t) \geq 0, (a \leq t \leq b).$$

In (P) and (D), the functions $f : I \times R^n \times R^n \rightarrow R$ and $g : I \times R^n \times R^n \rightarrow R^m$ are continuously differentiable; $x : I \rightarrow R^n$ is a piecewise smooth function (so that $\dot{x}$ is piecewise continuous) ; for each $t \ \varepsilon I$, $B(t)$ is a positive semidefinite $n \times n$ matrix with $B(\cdot)$ continuous on I ;$z : I \rightarrow R^n$ is piecewise smooth $\lambda : I \rightarrow R^m$ is a piecewise smooth function, considered as a row vector and the extended differentiation operator D is defined, for x piecewise smooth, by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\, ds; \qquad (8)$$

thus $D = \frac{d}{dt}$ except at discontinuities. The inequality (3)applies to each component of g. These hypotheses on the functions in (p) and (D) will be assumed in the theorems which follow.

The boundary conditions (2) may be replaced by x(a) = 0 = x(b) by a shift of origin in the space, x say, of functions x. Thus x may be taken as the vector space of piecewise smooth functions $x : I \rightarrow R^n$ for which x(a) = x(b), equipped with the norm $\| x \| = \| x \|_\infty + \| Dx \|_\infty$. This facilitates the proof of theorem 1; the original problem is recovered, as in [4], by the converse shift of origin. Note that (p) may be written as an optimal control problem, by substituting $u = \dot{x}$ and adjoining the differential equation Dx = u. It is well known from Pontryagin theory (see e. g.[4]) that an optimal solution will often require a discontinuity in u. The space x has been chosen to include such functions, with jump discontinuities in the derivative Dx.

CONDITIONS NECESSARY OR SUFFICIENT FOR OPTIMALITY

Theorem 1 (Necessary Conditions) : If (P) attains a (local) minimum at $x = \bar{x} \ \varepsilon \ x$, then there exist Lagrange multipliers $\tau \ \varepsilon \ R_+$ and piecewise smooth $\bar{\lambda} : I \rightarrow R_+^m$, not both zero, and also piecewise smooth $\bar{z} : I \rightarrow R^n$, satisfying for all $t \ \varepsilon \ I$:

$$\tau f_x\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big) + \bar{z}(t)^T B(t)) - \bar{\lambda}(t) g_x(t, \dot{\bar{x}}(t), \dot{\bar{x}}(t)) = D[\tau f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{\lambda}(t) g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))]; \qquad (9)$$

$$\bar{\lambda}(t)g\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right) = 0 \quad (10)$$

$$\bar{z}(t)^T B(t)\bar{z}(t) \le 1 \quad (11)$$

$$\bar{x}(t)^T B(t)\bar{z}(t) = (\bar{x}(t)^T B(t)\bar{x}(t))^{½} \quad (12)$$

Proof : The problem (P) may be written as

$$\underset{x \,\varepsilon\, X}{\text{Minimize}}\ \Phi_1(x) = F(x) + J_1(x)$$

Subject to,

$$G(x)\ \varepsilon\ S, \quad (13)$$

where

$$F(x) = \int_a^b f(t, x(t), \dot{x}(t))dt$$

$$J_1(x) = \int_a^b x(t)^T B(t)x(t))^{½}dt; \quad (14)$$

$G : X \rightarrow C(I, R^m)$ is given by $(\forall\, x\, \varepsilon\, X, \forall\, t\, \varepsilon\, I)$ $G(x)(t) = g(t, x(t), \dot{x}(t))$; and S is the convex cone of functions in $C(I, R^m)$ whose components are non-negative. Since G is Frechet differentiable Φ_1 is the sum of a Frechet differentiable function F and a nondifferentiable convex function J_1, and S is a convex cone with interior [5, Theorem 3] shows that necessary conditions for (13) to attain a local minimum at $x = \bar{x}$ are that Lagrange multipliers $\tau\ \varepsilon\ R_+$ and $\rho\ \varepsilon$ S exist, not both zero, satisfying

$$0\ \varepsilon\ \tau\,\partial\Phi_1(\bar{x}) + \partial(-\rho G)(\bar{x})\ ;\ \rho G(\bar{x}) = 0 \quad (15)$$

Where $\partial\Phi_1(\bar{x})$ and $\partial(-\rho G)(\bar{x})$are local subdifferentials [6],
given by $\partial\Phi_1(\bar{x}) = \{F'(\bar{x})\} + \partial J_1(\bar{x}); \partial(-\rho G)(x) = \{-\rho G'(\bar{x})\}$
where $R'(\bar{x})$ and $G'(\bar{x})$ are Frechet derivatives, and $\partial J_1(\bar{x})$ is the usual convex sub-differential. The cited theorems requires two convex sets to be weak $_*$compact – that is automatic for problem (13) – and further hypotheses on a constraint $-h(x)\ \varepsilon\ T$, which is absent in (13).
Since f(., ., .) is continuously Frechet differentiable it follows readily [4, page 16] that for x, v ε X.

$$F(x+v) - F(x) = \int_a^b [f_x(t, x(t), \dot{x}(t);\ v(t) + f_{\dot{x}}(t, x(t), \dot{x}(t))\ \dot{v}(t)]dt + 0(\| v \|)$$

Hence

$$F'(\bar{x})v = \int_a^b [f_x\left(t, x(t), \dot{x}(t)\right)v(t) + f_{\dot{x}}\left(t, x(t), \dot{x}(t)\right)\dot{v}(t)]dt \quad (16)$$

Assume now, subject to later verification, that $\rho\ \varepsilon\ S^*$ can be represented by a measurable function $\bar{\lambda} : I \rightarrow R^m$, satisfying with $\bar{\lambda}(t)$ as a row vector)

$$< \rho, v > = \int_a^b \bar{\lambda}(t)v(t)dt \quad (17)$$

Then $<\rho, v> \geq 0_b$whenever $v \geq 0$ requires $\bar{\lambda}(.) \geq 0$, Now for v ε X,

$$(\rho G)'(\bar{x})v = \int_a^b \bar{\lambda}(t)\left[g_x\left(t, x(t), \dot{x}(t)\right)v(t) + g_{\dot{x}}\left(t, x(t), \dot{x}(t)\right)\dot{v}(t)\right]dt \quad (18)$$

For t ε I, define $\theta_t : R^n \rightarrow R$ by $\theta_t(\xi) = (\xi^T B(t)\xi)^{½}$. From [7] the sub- differential

$$\partial\theta_t(\xi) = (w^T B(t) :\ w\ \varepsilon\ R^n, w^T B(t)w \le 1,$$

$$\theta_t(\xi) = w^T B(t)\ \xi\ \}\ (19)$$

Now $J_1(x) = \int_I \theta_t(x)dt$. From [6, Theorem 3] it follows that $y\ \varepsilon\ \partial J_1(\bar{x})$
if and only if a measurable function $\sigma : I \rightarrow R^n$ exists, for which (writing $\sigma(t)$as a row vector)

$$(\theta_t\ \varepsilon\ I)\ \sigma(t)\ \varepsilon\ \partial\theta_t(x) :\ (\forall\ v\ \varepsilon\ X) < y, v > = \int_a^b \sigma(t)v(t)dt \quad (20)$$

The cited theorem requires θ_t to be regular, which is fulfilled since θ_t is convex. Combining (19) and (20), $y\ \varepsilon\ \partial J_1(x)$if and only if a measurable function $\bar{z} : I \rightarrow R^n$ exists, such that

$$(\forall\ t\ \varepsilon\ I)\bar{z}(t)^T B(t)\bar{z}(t) \le \bar{z}(t)^T B(t)\bar{x}(t) = (\bar{x}(t)^T B(t)\bar{x}(t)^{½} \quad (21)$$

$$(\forall\ v\ \varepsilon\ X) < y, v > = \int_a^b \bar{z}(t)^T B(t)v(t)dt$$

Substitution of (16), (18) and (21) into (15) show that

$$(\forall\ v\ \varepsilon\ X) \int_a^b \{ [f_x\left(t, x(t), \dot{x}(t)\right) - \bar{\lambda}(t)g_x\left(t, x(t), \dot{x}(t)\right) + \bar{z}(t)^T B(t)]v(t) - \left[f_x\left(t, x(t), \dot{x}(t)\right) - \bar{\lambda}(t)g_{\dot{x}}\left(t, x(t), \dot{x}(t)\right)\right]\dot{v}(t)\}dt = 0 \quad (22)$$

Now the integrand of (22) has the form $P_1(t)v(t) + P_2(t)\dot{v}(t)$, where P_1 and P_2 are measurable functions, since f andg are continuously differentiable, $\bar{\lambda}$ is measurable, $\bar{x}$is piecewisesmooth, and $\bar{z}$ is measurable. Denote by P_3an indefinite integral ofP_1. Then integration by parts of

$$\int_a^b [P_1(t)v(t) + P_2(t)\dot{v}(t)]dt = 0,$$

using the boundary conditions $v(a) = 0 = v(b)$ from $v \varepsilon X$,

$$\text{show that} \int_a^b [P_2(t) - P_3(t)\dot{v}(t)dt = 0 \qquad (23)$$

and this must hold whenever $\dot{v}$ is replaced by a piecewise continuous function ζ for which $\int_a^b \zeta(t)dt = 0$. From [8, page 500-1, Lemma 2], it follows that $P_2(.) - P_3(.)$ is constant almost everywhere. (The cited Lemma assumes $P_2 - P_3$ is piecewise continuous, and deduces that it is constant ; the measurable extension is immediate). Hence, for almost all t ε [a, b], $P_2(.)$ is differentiable and satisfies $\dot{P}_2(t) = P_1(t)$. Substituting for P_1 and P_2 proves (9) for almost all t.Similarly (10) is proved from $\rho G(\bar{x})=0$ in (15) Although (17) is generally valid only if $\lambda(.)$ is a Schwarz distribution, the system of (9) and (10) is a linear first order ordinary differential equation for $\bar{\lambda}(.)$, given $\bar{x}(.)$ and $\bar{z}(.)$ and is therefore solvable for a piecewise continuous function $\bar{\lambda}(.)$. Consequently, from (9), $\bar{z}(.)$ may also be taken as piecewise smooth and then P_1 and P_2 are piecewise continuous. Hence $P_2(.)-P_3(.)$ is constant for all t, hence (9) and (10) hold for all t ε [a, b]. Also (21) gives (11) and (12).

Remarks 1: If B(t)=0 for each t ε I, then (P) and (D) reduce to the pair of variation problems considered by Mond and Hanson [1].

Theorem 1. gives Fritz John necessary conditions. Kuhn-Tucker conditions hold if also $\tau = 1$; the optimum $\bar{x}$ may then be called normal. Robinson condition [9] [4, page 150] is assumed for (13), namely

$$G(\bar{x}) + \{G'(\bar{x})v : v \varepsilon X\} - S \supset N_0,$$

a neighborhood of 0 in G(I, R^m), or if instead the Slater condition is assumed :

$$(\exists\, v \varepsilon X)\, G(\bar{x}) + G'(x)v \,\varepsilon\, \text{int}\, S. \qquad (24)$$

or equivalently if, for some v ε X and all t ε I.

$$g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + g_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)v(t) + g_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\dot{v}(t) > 0$$

Thus, if the Fritz John conditions (15) hold with $\tau = 0$, and (24) is assumed, the $0 \neq \rho \varepsilon S^*$, so $u \equiv G(\bar{x}) + G'(\bar{x})v \,\varepsilon\, \text{int}\, S$ satisfies both $\rho(u) > 0$, *and also* $\rho(u) = \rho G(\bar{u}) + \rho G'(\bar{x})v = 0 + 0$, in contradiction ; hence $\tau \neq 0$.

If (3) is generalized to g(t, x(t), ẋ(t)) ε T, where T is a closed convex cone in R^n, having interior points, then $\lambda(t) \varepsilon T^*$, the dual cone of T, in Theorem 1 [4]. Theorem 2 extends (P) to the problem (P_+), obtained by adjoining an equality constraint h (t, x(t), ẋ(t)) = 0 (a ≤ t ≤ b) to (P), where $h : I \times R^n \times R^n \to R^p$ is continuously differentiable. This extension is required for converse duality (Theorem 5).

Theorem 2 : If (P_+) attains a (local) minimum at $\bar{x}$, and if $h_x\left(., \bar{x}(.), \dot{\bar{x}}(.)\right)$ maps X onto a closed subspace of C (I, R^p), then there exist Lagrange multipliers $\tau \varepsilon R_+$, piecewise smooth $\bar{\lambda} : I \to R_+^m$, and
$q : I \to R^p$, not all zero, and also piecewise smooth $\bar{z} : I \to R^n$, satisfying, for all t ε I, (10), (11) and (12), and also (9), modified by the addition of an extra term

$$q(t)h_x(t, \bar{x}(t), \dot{\bar{x}}(t)) - D[q(t)h_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)]$$

on the right hand side. If $h_x(t, \bar{x}(.), \dot{\bar{x}}(.))$ is surjective, then τ and $\bar{\lambda}$ are not both zero.

Proof : Define $H : X \to C(I, R^p)$ by H(x)(t) = h(t, x(t), ẋ(t)) for x ε X, t ε I.

If $h_x\left(., \bar{x}(.), \dot{\bar{x}}(.)\right)$ is surjective, then H' ($\bar{x}$) is subjective, and then (15) is replaced by

$$\varepsilon\, \tau\, \partial\Phi_1(\bar{x}) + \partial(-\rho G)(\bar{x}) + \zeta H'(\bar{x})\,;\ \rho G(\bar{x}) = 0\,;$$

for τ and ρ not both 0, and some ζ in the dual space of C(I, R^p). This follows from [5, Theorem 3], nothing also that the smoothness requirements on f, g and h are fulfilled here for continuously differentiable functions.The remainder of the proof follows closely to that of Theorem 1 ; the details are hence omitted. If instead $h_x(., \bar{x}(.), \dot{\bar{x}}(.))$ has closed range but is not surjective, then H'($\bar{x}$) has closed range but is not subjective. Then there is nonzero ζ in the dual space of C (I, R^p) for which $\zeta H'(\bar{x}) = 0$; here τ and ρ may be chosen as zero.

Theorem 3 (Sufficient Condition) : Let f(t, ., .) and $-g(.,.,.)$ be convex functions, for each t ε I : let $(\bar{x}, \bar{\lambda}, \bar{z})$ satisfy the necessary conditions of Theorem 1, with $\bar{x}$ feasible for (P) and normal. Then $\bar{x}$ is optimal for (P).

Proof : For any x feasible for (P),

$$\Phi_1(x) - \Phi_1(\bar{x}) \geq \int_a^b \{f_x(t, \bar{x}(t), \dot{\bar{x}}(t))(x(t) - \bar{x}(t)) + (x(t)^T B(t) x(t))^{1/2} - (\bar{x}(t)^T B(t) \bar{x}(t))^{1/2}\}dt \text{ (since } f(t,.,.) \text{ is convex)}$$

$$\geq \int_a^b \{f_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right)\left(x(t) - \bar{x}(t)\right) + f_x((t, \bar{x}(t), \dot{\bar{x}}(t))(\dot{x}(t), \dot{\bar{x}}(t)) + (x(t) - \bar{x}(t))^T B(t) \bar{z}(t)\}dt$$

(by Schwarz inequality (11), (12))

$$= \int_a^b \{\bar{\lambda}(t) g_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + D[f_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) - \bar{\lambda}(t) g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))]\}(x(t)) - \bar{x}(t)dt + \int_a^b f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))(\dot{x}(t) - \dot{\bar{x}}(t))dt \text{ (by (9)}$$

$$= \int_a^b \bar{\lambda}(t)[g_x(t, \bar{x}(t), \dot{x}(t))(x(t) - \bar{x}(t) + g_{\dot{x}}\left(t, \left(\bar{x}(t), \dot{\bar{x}}(t)\right)\left(\dot{x}(t) - \dot{\bar{x}}(t)\right)\right] dt$$

(integrating by parts, using the boundary conditions (2), (5))

$$\geq \int_a^b \bar{\lambda}(t)\left\{g\big(t, x(t), \dot{x}(t)\big) - g\big(t, \bar{x}(t), \dot{\bar{x}}(t)\big)\right] dt$$

(since$-g(t,.,.)$ is convex and $\bar{\lambda}(t) \geq 0$)

$$= \int_a^b \bar{\lambda}(t) g\big(t, x(t), \dot{x}(t)\big) - 0 \text{ (by (10)}$$

$$\geq 0 \left\{ \text{by } \bar{\lambda}(t) \geq 0 \text{ and (3)} \right\}$$

Remark : If (3) is generalized to$g(t, x(t), \dot{x}(t)) \varepsilon T$, then $g(t,.,.)$ must be assumed T-convex [4, page 29] in Theorem 3.

DUALITY

Duality results will be proved, assuming that f and $-g$ are convex functions of (x, ẋ). From the proof of theorem 1, the problems (P) and (D) can be equivalently expressed as

$$\underset{x \,\varepsilon\, X}{\text{Minimize}}\ \Phi_1(x)$$

Subject to, $G(x)\ \varepsilon\ S$, (26)

And $\underset{y \,\varepsilon\, X, \rho}{\text{Maximize}}\ L(y, \rho) \equiv \Phi_1(y) - \rho G(y)$

subject to $\rho\ \varepsilon\ S^*, 0\ \varepsilon\ \partial L(y, \rho)$ (27)

where the sub differentialρ relates to L(., ρ) with ρ fixed.

Note that, for (D), $\psi_1(y, \lambda, z)$ = L(y, ρ), with (λ, z) constructed from ρ as in the Theorem 1.

Theorem 4 (Duality): Let f(t, ., .) and $-g(t,.,.)$ be convex functions, for each t ε I. If x ε C_p and $(y, \lambda, z)\ \varepsilon\ C_D$, then weak duality holds, thus $\Phi_1(x) \geq \psi_1(y, \lambda, z)$. If $\bar{x}$minimizes (P) and $\bar{x}$ is normal, then there exists $(\bar{x}, \bar{\lambda}, \bar{z})$ which maximizes (D), and$\Phi_1(\bar{x}) = \psi_1(\bar{x}, \bar{\lambda}, \bar{z})$.

Proof : Since f and $-g$ are convex functions of (x, ẋ) it follows that

L(., ρ) is convex ; and G is S-convex namely $\alpha G(x) + (1-\alpha)G(x') - (\alpha x + (1-\alpha)x')\ \varepsilon\ S$ whenever x, x' ε X and 0 <α < 1. Let x ε C_p and $(y, \lambda, z)\ \varepsilon\ C_D$. Then

$$\Phi(x) - \psi(y, \lambda, z) = \Phi(x) - L(y, \rho) \geq L(x, \rho) - L(y, \rho) \text{since } \rho G(x) \geq 0 \geq 0(x-y) \text{ since } 0\ \varepsilon\ \partial L(y, \rho), \quad (28)$$

Since x minimizes (P) and $\bar{x}$ is normal, the conclusions of Theorem 1 are satisfied by $(\bar{x}, \bar{\lambda}, \bar{z})$, with $\tau = 1$. Hence $(\bar{x}, \bar{\lambda}, \bar{z})\ \varepsilon\ C_D$, and

$$\Phi_1(\bar{x}) = \int_a^b [\, f(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B(t) \bar{x}(t))^{½}] dt$$

$$= \int_a^b [f(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B(t) \bar{z}(t) -$$

$$\bar{\lambda}(t) g(t, \bar{x}(t), \dot{\bar{x}}(t))] dt \text{ (by (12) and (10))}$$

$$= \psi_1(\bar{x}, \bar{\lambda}, \bar{z}) \text{ (29)}$$

From (29) and (28), $(\bar{x}, \bar{\lambda}, \bar{z})$ maximizes (D).

The weak duality property (28) can also be proved directly, without using the equivalent problem (27) by using results of [1]. Also, Theorem 4 may be proved, using (26) and (27). The convexity hypotheses in Theorem 4 may be weakened, to f pseudoconvex and $-g$ quasiconvex.

CONVERSE DUALITY

In this section, second derivatives of x are required. The space X must now be replaced by the smaller space X_2 of piecewise twice-differentiable functions x : I → R^n, for which x(a) = 0 =x(b), equipped with the norm$\| x \| = \| x \|_\infty + \| Dx \|_\infty + \| D^2 x \|_\infty$, defining D, as before. Problem (D) may be written in the form

$$\text{Minimize } -\psi(x, \lambda, z)$$

subject to $x(a) = \alpha, x(b) = \beta$

$$\theta\big(t, x(t), \dot{x}(t), \ddot{x}(t), \lambda(t), \dot{\lambda}(t), z(t)\big) = 0, z(t) B^T(t) z(t) \leq 1$$

$$\lambda(t) \geq 0 \ (a \leq t \leq b),$$

where $\theta \equiv \theta(t, x(t), \dot{x}(t), \ddot{x}(t), \lambda(t), \dot{\lambda}(t), z(t))$

$$= f_x\big(t, x(t), \dot{x}(t)\big) + z(t)^T B(t) - \lambda(t) g_x(t, x(t), \dot{x}(t))$$

$$-D\big[f_{\dot{x}}\big(t, x(t), \dot{x}(t)\big) - \lambda(t) g_{\dot{x}}\big(t, x(t), \dot{x}(t)\big)\big], (30)$$

and$\dot{x} = Dx$, $\ddot{x} = D^2 x$. A converse duality theorem will be proved, assuming a hypothesis that

$$\sigma(t)\theta_x - D(\sigma(t)\theta_{\dot{x}}) + D^2(\sigma(t)\theta_{\ddot{x}}) = 0 \text{ (31)}$$

$$\text{implies} \sigma = 0 \text{ (thus } \sigma(t) = 0 \text{ for } a \leq t \leq b \text{)}.$$

Consider $\theta(.,x(.),\dot{x}(.),\ddot{x}(.),\lambda(.),\dot{\lambda}(.),z(.))$ as a defining mapping $Q : X_2 \times \Lambda \times Z \to U$, where Λ is the space of piecewise differentiable functions λ, Z is the space of piecewise smooth functions Z, and U is a Banach space. A Fritz John theory may be applied to problem (D), such as Theorem 2, or (since (D), unlike (P), is a differentiable problem) Theorem 4.4.3 of [4] or the theorems of valentine [10]. But some restriction is then required on the equality constraint $\theta(.) = 0$, since infinite dimensional spaces are involved here [4], page (54) for a relevant counter-example. If suffices if the Frechet derivative Q' ≡ $[Q_x(\bar{x}, \bar{\lambda}, \bar{z}), Q_\lambda(\bar{x}, \bar{\lambda}, \bar{z}), Q_z(\bar{x}, \bar{\lambda}, \bar{z})]$ has (weak * closed range ([4 page, 59]).

Theorem 5 (Converse Duality) : Let (D) attain a (local maximum at $(\bar{x},\bar{\lambda},\bar{z})$, with $\bar{x}\ \varepsilon\ x_2$, $\bar{\lambda}$ and $\bar{z}$ piecewise smooth and let Q' have (weak *) closed range. Let f and g be twice continuously differentiable. Assume that the only piecewise smooth function σ satisfying (31) is the zero function. Then $\bar{x}$ minimizes (P), and the objective functions $\Phi(\bar{x})$ and $\psi(\bar{x},\bar{\lambda},\bar{z})$ are equal there.

Proof : Since $(x,\bar{\lambda},\bar{z})$ minimizes (D), with $\bar{x}\ \varepsilon\ X_2$, and Q' has (weak *) closed range, the Fritz John theorem (e.g. Theorem 2) shows that there exist Lagrange multipliers $\alpha\ \varepsilon\ R_+$, and piecewise continuous

$\mu : I\to R^n$, $\beta : I\to R_+$, $: I\to R_+^m$, not all zero, satisfying the condition :

$$\alpha[(f_x-\lambda(t)g_x)-D(f_{\dot{x}}-\lambda(t)g_{\dot{x}})+B(t)\bar{z}(t)]$$
$$-\mu(t)\theta_x+D(\mu(t)\theta_{\dot{x}})-D^2(\mu(t)\theta_{\ddot{x}})=0 \quad (32)$$
$$-\mu(t)B(t)+\alpha B(t)\bar{x}(t)-2\beta(t)B(t)\bar{z}(t)=0 \quad (33)$$
$$-\alpha g^T+\gamma(t)-\mu(t)\theta_\lambda+D(\mu(t)\theta_{\dot{\lambda}})=0 \quad (34)$$
$$\beta(t)(\bar{z}(t)^T B(t)\bar{z}(t)-1)=0 \quad (35)$$
$$\gamma(t)\bar{\lambda}(t)=0 \quad (36)$$

Here, for brevity, $f\equiv\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)$, $g\equiv g\left(t,x(t),\dot{x}(t)\right)$,

$f_x\equiv f_x\left(t,\bar{x}(t),\dot{\bar{x}}(t)\right)$, etc. θ is as in (30), with all derivatives evaluated at $x=\bar{x}$;$\dot{x}\equiv D_x$ and $\ddot{x}\equiv D^2x$. Note that the term $D^2(\mu(t)\theta_{\ddot{x}})$ in (32) is obtained using integration by parts, as in the proof of (23), with the boundary conditions $\mu(a)=0,\mu(b)=0$, adjoined to the differentiable equations (32), (33) so that the integrated parts vanish. Since f and g are twice continuously differentiable, θ is continuously differentiable.

From (32) and the constraint (6) of (D), μ must satisfy (31) with $\sigma=\mu$. The hypothesis shows that $\mu=0$. Suppose if possible, that $\alpha=0$. Then also $\gamma=0$ from (33). From (33), $\beta(t)B(t)\bar{Z}(t)=0$. Then from (35)

$\beta(t)=\beta(t)\bar{Z}(t)^T B(t)\bar{Z}(t)=0$.

Thus μ,α,β,γ are all zero, contrary to the Fritz John theorem. Hence α= 1 can be assumed.

Now (33) with $\alpha=1$ and $\mu=0$ gives $g=\gamma^T\geq 0$; so $\bar{x}$ is feasible for (P), and $\bar{\lambda}(t)g=0$ by (36). Since (33) implies equality in the Schwarz inequality

$$\bar{x}(t)^T B(t)\bar{z}(t)=(\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}(\bar{z}(t)^T B(t)\bar{z}(t))^{1/2} \quad (37)$$

From (35), $\beta(t)=0$ implying $B(t)\bar{x}(t)=0$ by (33) or $\bar{z}(t)^T B(t)\bar{z}(t)=1$. In either case, (37) gives

$$\bar{x}(t)^T B(t)\bar{z}(t)=(\bar{x}(t)^T B(t)\bar{x}(t))^{1/2} \quad (38)$$

$$\text{Then } \psi_1(\bar{x},\bar{\lambda},\bar{z})=\int_a^b[f(t,\bar{x}(t),\dot{\bar{x}}(t))+\bar{x}(t)^T B(t)\bar{z}(t)-\bar{\lambda}(t)g(t,\bar{x}(t),\dot{\bar{x}}(t))]dt=\int_a^b[f(t,\bar{x}(t),\dot{\bar{x}}(t))+(\bar{x}(t)^T B(t)\bar{x}(t))^{1/2}-0]dt$$
$$=\Phi_1(\bar{x})$$

RELATED PROBLEMS

As in [1] and [11], the duality results Can be extended to the corresponding problems (P1) omitting the boundary conditions (2), and (D1), with "natural boundary values" These problems are as follows.

$$\text{Primal (P1) : Minimize } \int_a^b[f(t,x(t),\dot{x}(t))+(x(t)^T B(t)x(t))^{1/2}]dt$$

Subject to

$$g(t,x(t),\dot{x}(t))\geq 0\ (a\leq t\leq b) \quad (39)$$

$$\text{Dual (D1) Maximize } \int_a^b [f(t,x(t),\dot{x}(t))+x(t)^T B(t)z(t)-\lambda(t)g(t,x(t),\dot{x}(t))]dt$$

Subject to

$$\lambda(t)\geq 0,(a\leq t\leq b)\ ;$$
$$f_x\left(t,x(t),\dot{x}(t)\right)+z(t)^T B(t)-\lambda(t)g_x\left(t,x(t),\dot{x}(t)\right)$$
$$=D[f_{\dot{x}}(t,x(t),\dot{x}(t))-\lambda(t)g_{\dot{x}}(t,x(t),\dot{x}(t))]\ (a\leq t\leq b)$$
$$z(t)^T B(t)z(t)\leq 1\ (a\leq t\leq b)$$
$$[f_{\dot{x}}(t,x(t),\dot{x}(t))-\lambda(t)g_{\dot{x}}(t,x(t),\dot{x}(t))]=0$$

when t = a and t = b (40)

The boundary conditions (40) are similar to "natural boundary conditions" in the calculus of variations ([12, page 206]).

The proof of weak duality given in Theorem 4 does not depend on the boundary conditions; hence weak duality applies to (P1) and (D1) assuming convex hypotheses. Alternatively, weak duality may be proved as in [1] using (40). To prove that $\Phi_1(\bar{x})=\psi_1(\bar{x},\bar{\lambda},\bar{z})$ holds also for (P1) and (D1), assume that $\bar{x}$ minimizes (P1). Then $\bar{x}$ also minimizes (P1), obtained by adjoining the constraints $x(a)=\bar{x}(a)$, $x(b)=\bar{x}(b)$ to (P1) ; hence the conclusions of Theorem 1 hold also for (P1). So it remains only to show that (40) holds when $x=\bar{x}$. In the notation of (23),

$$\int_a^b[\dot{P}_2(t)v(t)+P_2(t)\dot{v}(t)]dt=0 \quad (41)$$

Whenever $v(t)=x(t)-\bar{x}(t)$ and x ε X. Hence $[P_2(t)v(t)]|_a^b=0$.

Since now v (a) and v (b) are not fixed, $P_2(a)=0=P_2(b)$. Using (6), this proves (40) when $x=\bar{x}$. The proof of Duality theorem 4 then applies also to (p1) and (D1), assuming $\bar{x}$ is normal for (p1).

In particular, If (P1) and(D1) are independent of t, thus if f, g, h do not depend explicity on t, then these problems essentially reduce to the static cases of non-differentiable mathematical programs studied by Mond [2], namely

Primal (P2) Minimize $f(x) + (x^T Bx)^{½}$

Subject to

$$g(x) \geq 0;$$

Dual (D2) Maximize $f(x) + x^T Bz - \lambda^T g(x)$

Subject to

$$f_x(x) + z^T B = \lambda^T g_x(x), z^T Bz \leq 1, \lambda \geq 0.$$

It is noted, similarly to [1], that the hypotheses of Theorems 1, 3, 4, 5 reduce to the usual hypotheses for the static case in Mond [2]. Also the converse duality theorem for (D2) and (P2) does not require the hypothesis $\mu(a) = 0 = \mu(b)$.

REFERENCES

1. B. Mond and M. A. Hanson, "Duality for variational problems", J. Math. Anal. Appl. 18(1967), P. 355-364.
2. B. Mond, "A class of non-differentiable mathematical programming problems ", J. Math. Anal. Appl. 46(1974), P. 169-174.
3. B. D. Craven, Mathematical programming and control Hall, London (1978).
4. B. D. Craven and B. Mond, " Lagrangean conditions for quasi-differentiable optimization", proc. of 9th International symposium on mathematical programming, Budapest, (1976), in survey of Mathematical programming, A Prekopa (ed.) vol.1 P.177-19AkademaiKiado, Budapest and North – Holand, Amsterdam (1979).
5. F. H. Clarke, " Generalized Gradient of Lipschitz functional ", Adv. in Mathematics 40(1981), P. 52-67.
6. B.D. Craven and B. Mond, "Sufficient-Fritz John optimality conditions for Non-differentiable convex programming", J. Aust. Math. Soc. Series B 19(1976), P. 462-468.
7. R. Courant, Differential and Integral Calculus, Vol. Blackie, London and Edinburgh, 1936/1948.
8. S.M. Rohinson, " Stability theory for systems of inequalities ", Part II: Differentiable non-linear system ", SIAM J. Number. Anal. 13(1967) P. 497-513.
9. F. A. Valentine, "The problem of Lagrange with differential inequalities as added side constraints, contributions to the Calculus of variations 1933-1937 P. 407-448, University of Chicago Press (1937).
10. B. Mond and M. A. Hanson, "Symmetric duality for variational problems ", J. Math. Anal. Appl. 23(1968), P. 161-17
11. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. Wiley-Interscience, New York 1960.