THE SOLVABILITY OF GRAPH INEQUALITIES OVER DIRECTED GRAPHS

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Abstract: This paper is concerned with the solvability of inequalities or directed graphs $G(X)$ and $G_2(X)$ such that $G_1(X) \subseteq G_2(X)$. Graph inequalities have been studied survey papers dating back to the late 1970s by Capobianco (1979) and Cvetković and Smile (1977, 1979). The equalities and inequalities considered in these papers are more general in that they allow arbitrary operations on graphs such as complementation, tensor products, and squaring. In this paper we will show that the solvability of graph inequalities, i.e., having only one graph variable with one specified vertex, is decidable. This will follow from an (exponential) upper bound on the size of a minimal solution (if there is one).

Keywords : solvability of inequalities, Minimality, graph inequality, nondeterministic exponential time.

0. Introduction

This paper is concerned with the solvability of inequalities or directed graphs $G(X)$ and $G_2(X)$ such that $G_1(X) \subseteq G_2(X)$. Graph inequalities have been studied in the more general sense in which we allow such as complementation, tensor products and squaring. The more general question of which the graphs fulfill the condition $G^2 = \bar{G}$ (the complement of G) is still an open question, but it is known that this equation has an infinitely many solution. There is no connected, finite solution of the inequalities as noted in figure 1. We shall conclude the proof of this result below:

1. A simple graph inequality.

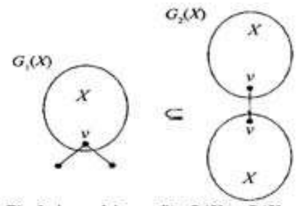

Fig. 1. A graph inequality $G_1(X) \subseteq G_2(X)$.

Consider the diagram in Figure 1.

Is there a solution to this inequality? More precisely, is there an undirected graph X with a vertex v such that if we construct a graph $G_2(X)$ by taking two copies of X and connecting v vertices by an edge, and a graph $G_1(X)$ by adding two new vertices to X and connecting them with v, then $G_1(X)$ occurs as a subgraph of $G_2(X)$?

A moment's reflection will show that the answer is yes: Take X to be a path of length two together with an isolated vertex v. What happens if we restrict ourselves to connected graphs? Again the answer is yes: Take a rooted infinite ternary tree and connect its root by an edge to a new vertex v. What about finite and connected graphs? the patient reader will ask. The answer in this case, is no, there is no finite, connected graph X fulfilling the inequality in Figure 1, and this is the main result of this section.

Theorem 1.1. There is no connected, finite solution of the Figure 1 inequality.

A simpler version of this theorem (for finite trees) was used by M. Schaefer [9, 10] to determine the computational complexity of the arrowing relation in graph Ramsey theory: deciding $F \to (T, K_n)$ is complete for the second level of the polynomial-time hierarchy (where F is a finite graph, T is a finite tree of size at least two, and K_n is the complete graph on n vertices).

Graph equations (more so than graph inequalities) have been studied for a while, and there are two survey papers dating back to the late 1970s [2, 5, 6]. The equalities and inequalities considered in these papers are more general in that they allow arbitrary operations on graphs such as complementation, tensor products, and squaring. Capobianco, Losi, and Riley [3], for example, showed that there are no (nontrivial) trees whose square is the same as their complement [4]. The more general question of which graphs fulfill $G^2 = \bar{G}$ is still open [1], but it is known that the equation has infinitely many solutions [3].

Before we begin the proof we introduce some standard notation [7]. We write $G = (V, E)$ for a graph G with vertex set $V = V(G)$ and an edge set $E = E(G)$. The edge between vertices $u, v \in V$ is written as (u, v). The order of a graph is defined as $|V(G)|$, and the size $|G|$ is defined as $|E(G)|$. A graph is finite if it has finite order and connected if there is a path between any two of its vertices.

Proof of Theorem 1.1. Let X be a minimal solution of the inequality. Denote the copies of X in $G_2(X)$ by X_i, $i = 1, 2$. An element of X is either its edge or vertex. Given an element x of X, we denote the corresponding element of X_i by x_i.

Let ϕ be the embed ling of $G_1(X)$ into $G_2(X)$. Clearly $(v_1, v_2) \in \mathrm{Im}\phi$, since otherwise $G_1(X)$ would map into X_1 or X_2. Assume that there is an edge $e \in X$ such that neither e_1 nor e_2 is in $\mathrm{Im}\phi$. Let Y be the connected component of $X-\{e\}$ containing v. From the connectedness of $G_1(X)$ it follows that $\mathrm{Im}\phi \subseteq G_2(Y)$. Now the restriction of ϕ to $G_1(Y)$ is an embedding of $G_1(Y)$ into $G_2(Y)$, contradicting the minimality of X.

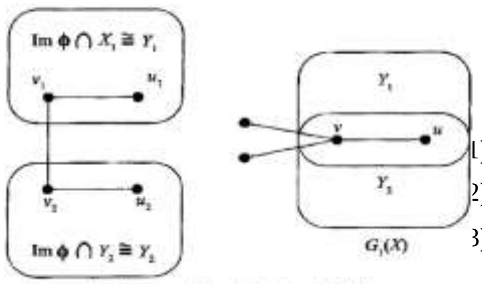

Fig. 2 : $\mathrm{Im}\phi$ and $G_1(X)$.

Thus for every $e \in X$ either e_1 or e_2 is in $\mathrm{Im}\phi$. Note that this implies that for every vertex $u \in X$ either u_1 or u_2 is in $\mathrm{Im}\phi$. Let Y_i be the subgraph of X corresponding to $\mathrm{Im}\ \phi \cap X_i$ (as a subgraph of X_i). Then for each $e \in X$ either $e \in Y_1$ or $e \in Y_2$. We know that

$$Y_1 \cup Y_2 = X, \quad (1)$$

$$|V(Y_1)| + |V(Y_2)| = |V(\mathrm{Im}\phi)| = |V(G_1(X))| = |V(X)| + 2, \quad (2)$$

$$|E(Y_1)| + |E(Y_2)| = |E(\mathrm{Im}\phi)| - 1 = |E(G_1(X))| - 1 = |E(X)| + 1. \quad (3)$$

The first equality in (3) follows from the fact that. $(v_1, v_2) \in \mathrm{Im}\phi$, but

$(v_1, v_2) \notin \mathrm{Im}\,\phi \bigcap (X_1, X_2)$. From (1), (2), (3) we conclude that $|V(Y_1 \bigcap Y_2)| = 2$ and $|E(Y_1 \bigcap Y_2)| = 1$ which implies that the intersection of Y_1 and Y_2 is a single edge f. We know that $v \in V(Y_1) \bigcap V(Y_2)$, and hence $f = (v, u)$ for some $u \in V(X)$. Figure 2 illustrates the situation.

Let a_i be the number of vertices from $V(Y_i) \setminus \{u, v\}$ which have degree 1 in X. Let b be 1 if u has degree 1 in X and 0 otherwise. The number of vertices of degree 1 in $G_1(X)$ is $a_1 + a_2 + b + 2$. The number of vertices of degree 1 in Im ϕ is at most $a_1 + a_2 + b + 1$. Hence Imϕ and $G_1(X)$ are not isomorphic, a contradiction.

2. DECIDABILITY OF GRAPH INEQUALITIES

We could now start considering all kinds of diagrams involving graphs, vertices, edges, and the subgraph relationship. How hard is it to settle these questions? In this section we will show that the solvability of graph inequalities of the type presented in the previous section, i.e., having only one graph variable with one specified vertex, is decidable. This will follow from an (exponential) upper bound on the size of a minimal solution (if there is one).

Let us formalize the question. A graph variable X with a set of specified vertices $v_1, ..., v_m$ represents an unknown finite, connected graph whose vertex set includes vertices $v_1, ..., v_m$. Given several graph variables $X_1, ..., X_n$ and a graph G, we can construct a graph term $G(X_1, ..., X_n)$ (called gterm) by taking several copies of each X_i and identifying some specified vertices of the copies with some vertices of G. Since we are working with connected graphs we require $G(X_1, ..., X_n)$ to be connected (for any assignment of connected graphs to $X_1, ..., X_n$). Note that G itself does not have to be connected and that if $G(X_1, ..., X_n)$ is connected for some assignment of connected graphs to $X_1, ..., X_n$, then it is connected for all assignments.

Given two such gterms $G_1(X_1, ..., X_n)$, $G_2(X_1, ..., X_n)$, we can ask whether there exists an assignment of connected finite graphs to the variables $X_1, ..., X_n$ such that $G_1(X_1, ..., X_n)$ is a subgraph of $G_2(X_1, ..., X_n)$. We call a question of this type a graph inequality.

For the rest of this section we will consider the simplest possible case of a graph inequality: only one variable, X, with one specified vertex v. Let $G_1(X)$ be gterm consisting of a connected graph H and a copy of X attached with v to each vertex of a multisubset $I = \{i_1, ..., i_\ell\}$ of vertices of H. Similarly construct $G_2(X)$ from a connected graph F and a multisubset $J = \{j_1, ..., j_k\}$ of vertices of F. The copy of X in $G_2(X)$ attached to $j_r (1 \le r \le k)$ is called $X_{(r)}$, and the copy of X in $G_1(X)$ attached to $i_r (1 \le r \le$ P) is called $X_{[r]}$. If there is only one copy of X in $G_1(X)$, we call it X.

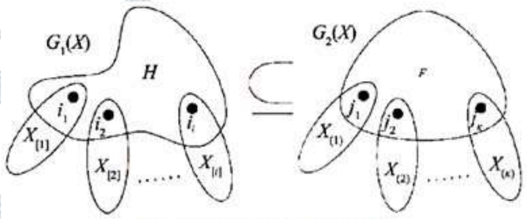

Fig. 3 : Inequality $G_1(X) \subseteq G_2(X)$.

Theorem 2.1. If the inequality in Figure 3 has a solution X, then it has a solution of size at most $|F|(1+k)^{|H|}$..

The upper bound on the size of a minimal solution is exponential in the size of the equality; hence to decide solvability we just have to test all graphs up to that size, something which can be done in nondeterministic exponential time (NEXP).

Corollary 2.2. The solvability of graph inequalities of the type in Figure 3 can be decided in NEXP.

We do not know the precise computational complexity of the decision problem. It is at least NP-hard, since we can ask whether a graph contains a clique.

At the core of the proof are Lemmas 2.5 and 2.7, which show that for a minimal solution to the graph inequality (if it exists) we can assume that all of the vertices of I are mapped to vertices of F. This reduces the problem to a simpler variant (namely, the images of vertices from I are prescribed) dealt with by Lemma 2.4 (based on the representation result of Lemma 2.3).

First we characterize solutions of inequalities (with prescribed mapping) where on the left-hand side there is only one copy of X and v has to map to a vertex w of F on the right-hand side.

If $w \in J$, then any connected graph is a solution. Now assume $w \notin J$. Let Σ be the alphabet consisting of the numbers 1, ... k. For each word a from Σ^* take a copy $F^{(\alpha)}$ of F. For every $\alpha \in \Sigma^*$ and $a \in \Sigma$ identify $w^{(\alpha a)}$ and $j_a^{(\alpha)}$. The resulting infinite graph is called F^∞ (see Figure 4).

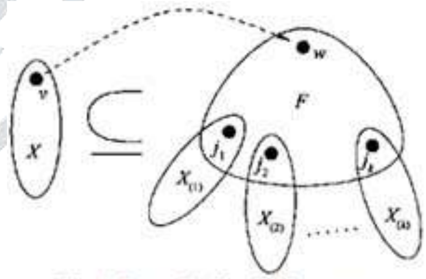

Fig. 4 : Inequality $X \subseteq G_2(X)$, $v \to w$.

Lemma 2.3. Assume that $w \notin J = \{j_1, \ldots, j_k\}$ Then the solutions of the inequality in Figure 5 are precisely the subgraphs X of F^∞ with $v = w^{()}$ such that

$$\begin{array}{l} \text{for any edge } e \text{ in } F, \text{ any } \alpha \in \Sigma^*, a \in \Sigma, \\ \text{if the edge } e^{(a\alpha)} \text{ is in } X, \text{ then } e^{(\alpha)} \text{ is in } X. \end{array} \qquad (4)$$

Proof. If X is a subgraph of F^∞ satisfying condition (4), then X is a solution of the inequality via mapping ϕ:

$$\phi(x^{()}) = x,$$
$$\phi(x^{(a\alpha)}) = x_{(a)}^{(\alpha)}.$$

If X is a solution of the inequality via mapping $\phi : X \to G_2(X)$, then define

$$Y^{()} = \phi^{-1}(F),$$
$$Y^{(a\alpha)} = \phi^{-1}(Y_{(a)}^{(\alpha)}),$$

where $Y_{(a)}^{(\alpha)}$ is the copy of $Y^{(\alpha)}$ in $X_{(a)}$ in $G_2(X)$. If e an edge of X with distance d from v, then it must map either to F or to some edge f in some $X_{(r)}$ which has strictly smaller distance from $v_{(r)}$ than d. Edges adjacent to v must be mapped to F, and hence they are in $Y^{()}$. By induction it follows that

$$X = \bigcup_{\alpha \in \Sigma} Y^{(\alpha)}.$$

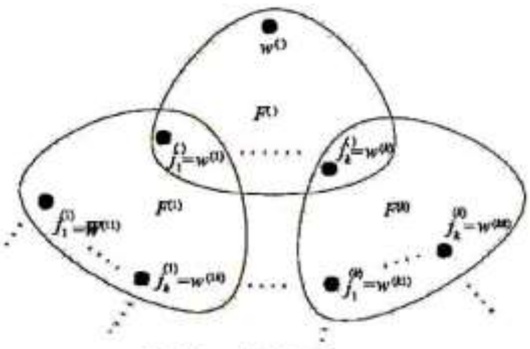
Fig. 5 : Inequality F^∞.

Clearly $Y^{(\alpha)}$ is a snhgraph of F via $\phi^{|\alpha|+1}$ for any $\alpha \in \Sigma^*$. The element of $Y^{(\alpha)}$ corresponding to $x \in F$ is called $x^{(\alpha)}$. By induction it follows that $w^{(\alpha a)} = j_{(a)}^{(\alpha)}$ for any $\alpha \in \Sigma^*$. From the definition of Y's, if $e^{(a\alpha)}$ is in X, then the edge $e^{(\alpha)}$ is also in X for any $\alpha \in \Sigma^*, \alpha \in \Sigma$. Hence X is a subgraph of F^∞ satisfying (4).

Solving systems of simple graph inequalities is useful in solving more complicated inequalities.

Lemma 2.4. If a system of inequalities with prescribed mappings

$$H_1 \subseteq X, h_1 \to v; ...; H_m \subseteq X, h_m \to v, \quad (5)$$

$$X \subseteq F_1(X), v \to w_1; ...; X \subseteq F_n(X), v \to w_n \quad (6)$$

has a solution, then it has a solution of size at most $|F_1|(1+k_1)^M$, where k_1 is the number of copies of X in F_1 and $M := \max\{|H_1|,...,|H_m|\}$, assuming that the graphs $H_1,...,H_m$ are connected.

Proof. Let X be a minimal solution of the system. Let e be an edge of X whose distance d from v is maximal. Assume that $d > M$. If we remove the edge e, then $X' = X - \{e\}$ still satisfies inequalities (5), because no edge of any $H_i(1 \le i \le m)$ can map to e. If X satisfies the inequality in Figure 5 for $F = F_i$ $(1 \le i \le n)$, then by Lemma 2.3 it is a subgraph of F^∞ with $v = w^{( )}$ and it satisfies condition (3.4). Let $e = f^{(\alpha)}$. Clearly X' is also a subgraph of F^∞ and the condition is still satisfied, because dist(v, $f^{(a\alpha)}$) > dist(v, $f^{(\alpha)}$) and hence $f^{(a\alpha)} \notin X'$ for any $a \in \Sigma$. Therefore X' satisfies inequalities (6), a contradiction to the minimality of X.

Thus dist(v, e) $\le M$. The size of the subgraph of F_1^∞ consisting of edges within distance M from v is bounded by $|F_1|(1+k_1)^M$.

Now we return to the inequality in Figure 3.

Lemma 2.5. If there is more than one copy of X on the left side of the inequality in Figure 3, then every $i_r = v_{[r]}(1 \le r \le \ell)$ must map to a vertex of F.

Proof. Suppose, for example, that i_1 maps into some $X_{(r)} - \{j_r\}$. Let P be a path from i_1 to i_2. Graphs $X_{[1]}$ and $X_{[2]} \cup P$ share only vertex i_1. Hence the image of at least one of them does not contain j_r and since j_r is a cutvertex of G_2, that image must be contained in $X_{(r)} - \{j_r\}$, which is impossible, since there are more vertices in $X_{[1]}$ or in $X_{[2]} \cup P$ than in $X_{(r)} - \{j_r\}$.

Lemma 2.6. If X is a solution of the inequality in Figure 3 via mapping $\psi : G_1(X) \to G_2(X)$, then there exists a mapping $\phi : G_1(X) \to G_2(X)$ such that $\phi(i) = \psi(i)$ and as many copies of X in i as possible are mapped to copies of X in $\phi(i)$ for every $i \in I$.

Proof. Consider a bipartite graph B with partitions I and J where i_r is connected to j_s if and only if $\psi(i_r) = j_s$. Without loss of generality assume that $\{(i_r, j_s); 1 \le r \le t\}$ is a maximal matching of B.

We need to show that there exits ϕ such that $X_{[r]}$ maps to $X_{(r)}$ for $1 \le r \le t$. Let Y^1, …, Y^q be the connected components of $X - \{v\}$. Let ϕ be a mapping such that

$$\sum_{r=1}^{t}\sum_{j=1}^{q}\left|\phi\left(Y_{[r]}^j\right)\cap Y_{(r)}^j\right| \quad (7)$$

is maximal. If for some r, j,

$$\phi\left(Y_{[r]}^j\right) \ne Y_{(r)}^j,$$

then clearly $\phi\left(Y_{[r]}^j\right)\cap Y_{(r)}^j \ne \emptyset$; otherwise $\phi\left(Y_{[r]}^j\right)$ would have to contain j_r. Now we can change ϕ in such a way that $Y_{[r]}^j$ will be mapped to $Y_{[r]}^j$ and $\phi^{-1}\left(Y_{(r)}^j\right)$ will be mapped to $\phi\left(Y_{[r]}^j\right)$. This increases the value of (7), a contradiction. Hence ϕ maps $X_{[r]}$ to $X_{[r]}(1 \le r \le t)$.

We prove an analogue of Lemma 2.5 for inequalities where X occurs only once on the left-hand side of the inequality in Figure 3.

Lemma 2.7. If the inequality in Figure 6 has a solution, then it has a solution X via a mapping ϕ which maps $v = i_1$ to a vertex of F.

Proof. Suppose that there is no solution of the inequality in Figure 6 such that v maps to a vertex of F, but there is a solution in which v maps into a vertex of $X_{(1)} - \{j_1\}$. Then clearly the inequality in Figure 7 with the additional condition that v must map to some $u \in X_{(1)}$ has a solution (see Figure 7).

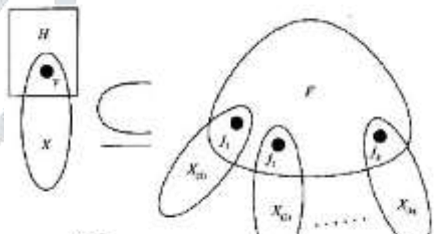

Fig 6

If $u = j_1$, then by Lemma 3.2.6 there is ϕ such that X is mapped to $X_{(1)}$. Therefore, we can replace K_∞'s in the inequality in Figure 7 by $K_{[H]}$'s, since only H is mapped to $G_2(X) - X_{(1)}$. This, however, implies that $X = K_{[H]}$ is a solution of the inequality in Figure 6 in which v maps to a vertex of F, a contradiction.

Thus $u \ne j_1$ for every solution of the inequality in Figure 7. Let X be a minimal solution of this inequality. Graphs H and X share only v; moreover j_1 is a cutvertex of G_2 and hence either H or X must be mapped inside $X_{(1)} - \{j_1\}$. Since the latter is not possible, H must be. mapped inside $X_{(1)} - \{j_1\}$.

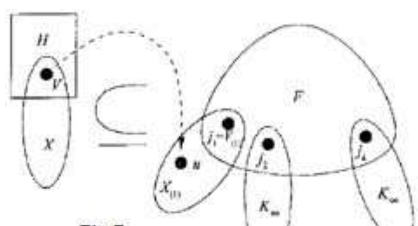

Fig. 7

Now let $Y = \phi^{-1}(X_{(1)}) \cap X$ and $Z = \phi^{-1}\left(G_2(X) - \left(X_{(1)} - \{j_1\}\right)\right)$.

The common vertex of Y and Z is called $q = \phi^{-1}(j_1)$. The inequality in Figure 7 implies the inequalities in Figure 8.

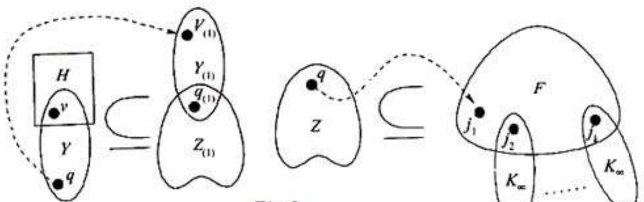

Fig. 8

The second inequality follows directly from the definition. To see the first inequality, note that the graph on the left-hand side is a subgraph of $X_{(1)}$ with q mapping to $j_{(1)}$, and that by definition of Y and Z the right-hand side contains $X_{(1)}$ with $j_{(1)}$ of $X_{(1)}$ mapping to v of Y.

If in the first inequality v was mapped outside of $Y_{(1)}$, then the shortest path from q to v would have to map to a longer path, which is not possible. Hence v maps inside $Y_{(1)}$. Combining the two inequalities in Figure 8, we get that Y satisfies the inequality in Figure 7. This contradicts the minimality of X.

We can now complete the proof of Theorem 2.1 by showing a bound on the size of a minimal solution (if there is one) of graph inequalities with one variable and one specified vertex.

Proof of Theorem 2.1. From Lemmas 2.5 arid 2.7 it follows that we need to consider only solutions in which every i_r ($1 \le r \le$ P) maps to a vertex of F. For each such mapping ϕ, using Lemma 3.2.6, we can assume that if $i \in I$ maps to a vertex $j \in J$, then as many copies of X in i as possible map to copies of X in j.

Let

$$G_1'(X) \subseteq G_2'(X), \quad v = i_1 \to \phi(i_1), \ldots, i_\ell \to \phi(i_\ell) \tag{8}$$

be the inequality with prescribed mappings obtained by removing those $X_{[r]}$'s and $X_{[r]}$'s which are already taken care of by Lemma 2.6. Notice that now no ($1 \le r \le$ P′) maps to a $j_s'(1 \le s \le k')$.

Let X' be a solution of (8) with mapping ψ. If $\psi\left(X'_{[r]}\right) \cap X'_{[s]} \neq \emptyset$, then some vertex from $X'_{[r]} - \{i'_\ell\}$ must map to j'_s. Since j'_s is a cut vertex, no other part of $G_1'(X')$ can map to $X'_{[s]}$. If for each $X'_{[s]}$, $1 \le r \le$ P′, and H we take the set of objects (edges and $X'_{[s]}$'s) to which it is mapped, then these sets are disjoint.

There are only finitely many partitions of the objects of $G_2'(X)$ into P + 1 disjoint sets. For each such partition we get a system of inequalities with prescribed mappings as in Lemma 2.4, which has a solution of size at most $|F|(1+k)^{|H|}$ (if it has one).

Note that by using previous lemmas we can easily prove Theorem 1.1. If there was a solution of the inequality in Figure 1, then by Lemma 2.7 there is a solution such that v from $G_1(X)$ maps to one of the v's in $G_2(X)$. By looking at the degree of v's we see that this is not possible.

We conclude this section with a technical result that allows us to combine several inequalities with prescribed mappings.

Lemma 2.8. For any system of inequalities with prescribed mappings

$$H_1 \subseteq X, h_1 \to v; \ldots; H_m \subseteq X, h_m \to v,$$
$$X \subseteq F_1(X), v \to w_1; \ldots; X \subseteq F_n(X), v \to w_n,$$

there is a single inequality which has the same set of solution as the system.

Proof. Consider the inequality in Figure 9.

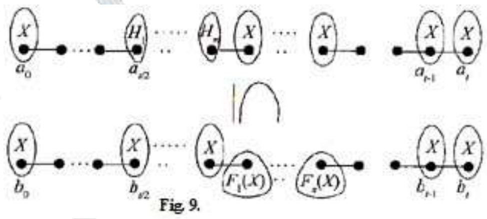

Fig. 9.

By Lemma 2.5; a_0 and a_t have to map to F. Clearly the a_0, a_t path of H in $G_1(X)$ has to map to a path in F in $G_2(X)$. If $t > 2(m + n + \max\{F_1, \ldots, F_n\})$, then the only path of length t in F is the b_0, b_t path. It follows that a_i maps to $b_i (0 \le i \le t)$ because u_{t-1} cannot map to a_l. Hence X is a solution of the inequality in Figure 9 if and only if it is a solution of the system.

REFERENCES

[1] Baltić, V.; Simić, S. K. and Tintor, V. (1994) : Some remarks on graph equation $G_2 = \bar{G}$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 5, p. 43-48.

[2] Capobianco, M. F. (1979) : Graph equations, Ann. New York Acad. Sci., 319, p. 114-118.

[3] Capobianco, M. and Kim, S.-R. (1995) : More results on the graph equation $G_2 = \bar{G}$, in Graph Theory, Combinatorics, and Algorithms. Wiley, New York, p. 617-628.

[4] Capobianco, M. F.; Losi, K. and Riley, B. (1989) : $G_2 = \bar{G}$ has no nontrivial tree solutions, Ann. New York Acad. Sci., 555, p. 103-105.

[5] Cvetković, D.M. and Smile, S.K. (1977) : Graph equations, in Contributions to Graph Theory and Its Applications, Technische Hochscliule Ilmenau, Ilmenan, Germany, p. 40-56.

[6] Cvetković, D.M. and Smile, S.K. (1979) : A bibliography of graph equations, J. Graph Theory, 3, pp. 311-324.

[7] Diestel, R. (1997) : Graph Theory, Grad. Texts in Math. 173, Springer-Verlag, New York.

[8] Schaefer, M. (1999) : Completeness and Incompleteness, Ph.D. thesis, University of Chicago, Chicago.

[9] Schaefer, M. (2001) : Graph Ramsey theory and the polynomial hierarchy, J. Comput. System Sci., 62, p. 290-322.