A Study of Assignment Problem and its Applications

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Abstract: Classic assignment problem is special case of linear programming problem. This is generally made on one to one basis. This paper is survey of the variations of the assignment problem. Assignment problems involve optimally matching the elements of two or more sets, where the dimension of the problem refers to the number of sets of elements to be matched. The intention here is to identify what these variations are and to make it easier for a researcher trying to develop some variation of the assignment problem for a particular application.

Keywords: Assignment problem, Hungarian method, linear programming problem,

I. INTRODUCTION

Linear programming deals with the optimization of a function of variables known as objective function, subject to a set of linear equations and/or inequalities known as constraints. Linear programming can be used for optimizations if there must be a well defined objective function which is to be either maximized or minimized and which can be expressed as a linear function of decision variables. Using linear programming allows researchers to find the best, most economical solution to a problem within all of its limitations, or constraints. Many fields use linear programming techniques to make their processes more efficient. These include food and agriculture, engineering, transportation, manufacturing and energy. Transportation model deals with the transportation of a product available at several sources to number of different destinations.

Linear programming technique may be used in assignment problem where the available resources that may be in the form of manpower, machines, tools, etc are to be allocated to varies jobs in the best way that minimizes the total cost or maximizes the total profit. The original version of the assignment problem is discussed in almost every textbook for an introductory course in either management science/operations research or production and operations management. Assignment problem is well structured linear programming problem, in which number of jobs (task) is equal to number of persons (facilities).Thus the objective of the problem is how the assignment should be made to achieved allocation. The most famous specific algorithm for the assignment problem is the 'Hungarian Algorithm' created by Kuhn[31]. This is named after the Hungarian mathematician König, who in 1916 proved a theorem necessary for the development of this method.

It is worth noting that the classic AP is mathematically identical to the weighted bipartite matching problem from graph theory, so that results from that problem have been used in constructing efficient solution procedures for the classic AP.

All classes of the assignment problem have the following characteristics:
-The overall objective is to minimize the cost of assigning tasks.
-Binary decision variables are used to indicate when a task is completed.
- Each task can only be assigned once.
- Functions must be linear, or have the ability to be converted into a linear nature.

II. LINEAR PROGRAMMING PROBLEM[37]

Linear programming (LP) can be defined as' a mathematical method of determining an optimum assignment of interdependent activities, given the availability of resources'. LP models may be constructed for various reasons. LP techniques are used in a range of industries for example agricultural, economics and management, transportation, petrochemicals, industrial industries, military operations, public and financial sectors, All LP problems involve optimization. Optimization requires the model to contain key objects. These are data or parameters variables (continuous, semi-continuous, free, binary, integer) constraints (equalities, inequalities) objective function. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources.

In 1947, George Dantzig found out this technique for solving military planning problem. He also developed the most powerful mathematical tool know as "simplex method" to solve linear programming problem. Linear programming is the most widely used technique in business, industries, and so many other fields of allotment and management decisions. Transportation problem and assignment problem are important application of LPP.

2.1 Assignment problem is a variant form of transportation problem.

The assignment problem is special case of the transportation problem with two characteristics. First, the pay off matrix for the problem would be square, and, second, the optimal solution to the problem would always be such that there would be only one assignment in a given row or column of the pay off matrix. In which the objectives is to assign a number of origins to the equal number of destinations at a minimum cost. For example, the assignees might be employees who need to be given work assignments. Assigning people to jobs is a common application of the assignment problem. However, the assignees need not be people. They also could be machines, or vehicles, or plants, or even time slots to be assigned tasks. To fit the definition of an assignment problem, these kinds of applications need to be formulated in a way that satisfies the following assumptions.

(i) The number of assignees and the number of tasks are the same. (This number is denoted by n).
(ii) Each assignee is to be assigned to exactly one task.
(iii) Each task is to be performed by exactly one assignee.
(iv) There is a cost c_{ij} associated with assignee i (i = 1, 2, . . . , n) performing task j(j= 1, 2, . . . , n).
(v) The objective is to determine how all n assignments should be made to minimize the total cost.

2.2 Formulation of Assignment Problem[34]

The decision problem becomes complicated when a number of resources are required to be allocated and there are several activities to perform. The decision problem can be formulated, and solved, as mathematical programming problem.
The Assignment problem can be stated in the form of n x n matrix, [C_{ij}] called the cost matrix, where C_{ij} assigning i-th job to j-th person.

Person

Job

	1	2			J			n
1	C_{11}	C_{12}			C_{1j}			C_{1n}
2	C_{21}	C_{22}			C_{2j}			C_{2n}
I	C_{i1}	C_{i2}			C_{ij}			C_{in}
N	C_{n1}	C_{n2}			C_{nj}			C_{nn}

A general solution method for algebraic linear assignment problems: Minimize the total cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} cij\, xij$$

Where xij = $\begin{cases} 1, if\ i-th\ person\ is\ assigned\ to\ the\ j-th\ job \\ 0, if\ i-th\ person\ is\ not\ assigned\ to\ the\ j-th\ job \end{cases}$

Subject to conditions

(i) $\sum_{i=1}^{n} xij = 1\ ,\ j = 1,2 \ldots .n$

Which means that only one job is done by the i-th person,i=1,2…n

(ii) $\sum_{j=1}^{n} xij = 1, i = 1,2 \ldots .n$

Which means that only one person should be assigned to the j-th job,j=1,2….n

2.3 Method for solving an assignment problem[34]

The evaluation of costs for all allocations will take a large time. Thus there is a need to develop an easy method for the solution of assignment problem. The well-known Hungarian method developed by Kuhn and published in 1955 [31] is recognized to be the first practical method for solving the assignment problem.
Hungarian Method for solving a minimal assignment problem;

I. Subtract the minimum element of each row in the cost matrix [c_{ij}] from every element of the corresponding row.
II. Subtract the minimum element of each column in the reduced matrix obtained in the step 1from every element of the corresponding column.

III. (a) Starting with row 1 of the matrix obtained in step II, examine rows successively until a row with exactly one zero element is found, mark [] at this zero , as an assignment will be made there. Mark[x] at all other zeros in the column to show that they cannot be used to make other assignments. Proceed in the way until the last row is examined.
(b) After examining all the rows completely proceed similarly examining the columns, examine the columns starting with column 1 until a column containing exactly one unmarked zero is found, mark [] at this zero and cross at all zero of the row in which []is marked . Proceed in this way until the last column is examined
(c) Continue these operations (a) and (b) successively until we reach to any of the two situations. (i) All the zeros are marked or crossed. Or (ii) the remaining zeros lies at least two in each row and column
in case (i),we have a maximal assignment and in case (ii) still we have some zeros to be treated for which we use the trial and error method to avoid the use of highly complicated algorithm. Now there are two possibilities;
(i) If it has an assignment in every row and every column, then the complete optimal assignment is obtained.
(ii) If it does not contain assignment in every row and every column, one has to modify the cost matrix by adding or subtracting to create some more zeros in it. For this proceed to the next step IV.
IV. When the matrix obtained in the step 3 does not contain assignment in every row and every column then we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. For this the following procedure is adopted:
(i) Mark (√) all rows for which assignment has not been made.
(ii) Mark (√) column which have zeros is marked rows.
(iii) Mark (√) rows which have assignment in marked columns.
(iv) Repeat step (ii) (iii) until the chain of marking ends.
(v) Draw minimum number of lines through unmarked rows and through marked columns to cover all the zeros. This procedure will yield the minimum number of lines that will pass through all zeros.
V. Select the smallest of the elements that do not have a line through them subtract it from all the elements that do not have a line through them, add it to every element that lies at the intersection of two lines and leave the remaining elements of the matrix uncha nged.
VI. At the end of step of V numbers of zeros are increased in the matrix than that in step III and to obtain the desired solution.

Theorem-1 (Reduction theorem) [34]: If, in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix [cij], then an assignment which minimize the total cost for one matrix, also minimizes the total cost for the other matrix.
Or
Mathematically the theorem may be stated as follows:

If xij=Xij, minimizes Z=$\sum_{i=1}^{n}\sum_{j=1}^{n} cij\,xij$ over all xij s.t.

$$\sum_{i=1}^{n} xij = 1 = \sum_{j=1}^{n} xij \text{ and } xij \geq 0$$

Then xij=Xij also minimizes Z'=$\sum_{i=1}^{n}\sum_{j=1}^{n} c'ij\,xij.$

Where c'ij=c'ij$\pm ai \pm bj$.
ai,bj are constants,i=1,2,3…..n; j=1,2,3…..n.

Proof: we have

$$Z' = \sum_{i=1}^{n}\sum_{j=1}^{n} c'ij\,xij$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (cij \pm ai \pm bj).xij$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} cij\,xij \pm \sum_{i=1}^{n}\sum_{j=1}^{n} ai\,xij \pm \sum_{i=1}^{n}\sum_{j=1}^{n} bj\,xij$$
$$= Z \pm \sum_{i=1}^{n} aij \sum_{j=1}^{n} xij \pm \sum_{j=1}^{n} bij \sum_{i=1}^{n} xij$$
$$= Z \pm \sum_{i=1}^{n} aij.1 \pm \sum_{j=1}^{n} bij.1.$$

$$= Z \pm \sum_{i=1}^{n} aij \pm \sum_{j=1}^{n} bij.$$

Since $\sum_{i=1}^{n} ai, \sum_{j=1}^{n} bj$ are independent of xij, it follows that Z' is minimized when Z is minimized.

2.4 Numerical example [34]
Five men are available to do five different jobs. From past records, the time (in hour) that each man takes to do each job is known and is given in the following table:
Job

Man

	A	B	C	D	E
1	1	3	2	3	6
2	2	4	3	1	5
3	5	6	3	4	6
4	3	1	4	2	2
5	1	5	6	5	4

SOLUTION: Step 1. Subtract the smallest element of each row from every element of the corresponding now, we get the following matrix:

	A	B	C	D	E
1	0	2	1	2	5
2	1	3	2	0	4
3	2	3	0	1	3
4	2	0	3	1	1
5	0	4	5	4	3

Step 2.Subtract the smallest element of each column from every element of the corresponding column ,we get the following matrix:

	A	B	C	D	E
1	0	2	1	2	4
2	1	3	2	0	3
3	2	3	0	1	2
4	2	0	3	1	0
5	0	4	5	4	2

Step3.Giving the zero assignment in usual manner and get reduced matrix

		A	B	C	D	E
1	[	0]	2	1	2	4
2		1	3	2	[0]	3
3		2	3	[0]	1	2
4		2	[0]	3	1	0
5		$\cancel{0}$	4	5	4	2

Since row 5 and column 5 have no assignment we proceed to the next step Step 4.The minimum numbers of lines drawn in the usual manner are 4.

Step5. Now the smallest of the elements that don not contains line through them is 1. Subtracting this element from the elements that do not have a line through them adding to every element that lies at the intersection of two lines and leaving the remaining elements unchanged,

	A	B	C	D	E
1	0	1	0	1	3
2	2	3	2	[0]	3
3	3	3	[0]	1	2
4	3	[0]	3	1	0
5	[0]	3	4	3	1

Step6. Again repeating the step 3 we make the zero assignments in matrix and see that even now the row 1 and column 5do not contain any assignments .Therefore we again repeat step 4 of drawing lines.

Step7. According to our usual manner the minimum number of lines drawn is 4 Step 8.Again repeating step5, we get following matrix.

	A	B	C	D	E
1	[0]	0	0	0	2
2	3	3	2	[0]	3
3	3	2	[0]	0	1
4	4	[0]	4	1	0
5	0	2	4	2	[0]

Step 9.Repeating the step 3 we make the zero assignments and get the following option assignments, 1→A,2→D,3→C,4→D,5→E.

So minimal assignment: 1+1+3+1+4=10.

2.5 Application of Linear Assignment Problem

A more practically useful application of the assignment problem is [3] the purpose of this research is to show the effectiveness of the assignment model in the University of Port Harcourt. Here n jobs to be performed with n companies and the problem are how to optimally assign these jobs to different companies involved. Six projects were to be undertaken in University of Port Harcourt. These projects were put on tender and six companies tendered in their bids. Then, the objective function is assigning the different jobs to different companies is to find the optimal assignment that will minimize the total time taken to finish all jobs by the companies. Engineers also use linear programming to help solve design and manufacturing problems. For example, in airfoil meshes, engineers seek aerodynamic shape optimization. This allows for the reduction of the drag coefficient of the airfoil. Linear programming therefore provides engineers with an essential tool in shape optimization.

The assignment problem is commonly used to aid different types of timetable scheduling. This is because it is easy to represent timetabling as a 0-1 assignment of activities to resources. Kanjana Thongsanit [8] published a research article for Classroom Assignment Problem for a University. The purpose of the study is to develop mathematical model and design methods to solve the problem. Excel's Premium Solver is applied in this study. It was found that Excel's Premium Solver can solve this classroom allocation problem with the process time in seconds.

Assignment problem may be used in agriculture planning of allotting lands for different crops and the distribution of available resources such as fertilizer, labor, water supply etc so that the total revenue may be maximized. Another use of assignment problem is described in article [32] this paper solves the problem of agriculture using R-Software. This paper is concerned with the special class of allocation problems, where the objective is to find optimal assignment of the number of paddocks to the number of crops used.

III. SPECIAL CASE IN CLASSIC ASSIGNMENT PROBLEM

3.1 Maximization of assignment problem[37]

Sometimes an assignment problem in which the objective functions is to be maximized. Such maximization problem may be solved by converting it to minimization problem. This transformation may be done in either of the following two ways: a) by subtracting all the elements from the largest element of the matrix. b) By multiplying the matrix elements by-1.

Maximization assignment problem can be converted to minimization assignment problem. Many methods are available for this problem. [2] Introduced a direct method for solving maximization problems. The method finds its applicability in all of an assignment problem and its kind.

3.2 Balanced and un balanced assignment problem[Non-square matrix][35]

There are many variations in assignment problem. An assignment problem is called an unbalanced assignment problem whenever the number of tasks is not equal to number of facilities. Since the Hungarian method of solution requires a square matrix, for such problem we add dummy rows or columns with costs of zero to the given matrix to make it a square matrix. One may also use non-zero costs for assignments using dummy tasks or agents to reflect differences based on which agents or tasks are not assigned.

3.3 Prohibited assignment problem[35,37]

Sometimes due to certain reason, technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problem can be solved by assigning a very heavy cost (infinite cost) to the corresponding cell. Such a job will then be automatically excluded from further consideration (making assignments). When a problem has a unique optimal solution, it means that no other solution to the problem exists which yields the same objective functions value as the one obtained from the optimal solution derived. On the other hand, in a problem with multiple optimal solutions, there exists more than one solution which all are optimal and equally attractive.

3.4 Travelling salesman problem as Assignment problem[1]

Travelling salesman problem is very similar to assignment problem expect that in the former there is an additional restrictions that a salesman who starts from his home city, visit each city only once and return to his home city. If the solution to the assignment does not satisfy the additional restriction, then after solving the problem by assignment technique, we use the method of enumeration.

Example:[37] Given the following matrix of set-up costs, show how to sequence production so as to minimize set-up cost per cycle:

	A	B	C	D	E
A	_	2	5	7	1
B	6	_	3	8	2
C	8	7	_	4	7
D	12	4	6	_	5
E	1	3	2	8	_

Solution: Reduce the cost matrix and make assignments in rows and columns having single zeros.

	A	B	C	D	E
A	_	1	3	6	[0]
B	4	_	[0]	6	0
C	4	3	_	[0]	3
D	8	[0]	1	_	1
E	[0]	2	0	7	_

The optimal assignment is A→E, E→A, B→C, C→D and D→B with minimum cost of 13.

But solution does not provide travelling salesman problem. Now try to find new solution which satisfies this extra restriction. The minimum nonzero element is 1, consider 1 as minimum element and try to making assignment.

The resulting solution will be A→B, B→C, C→D, D→E and then E→A with cost of 15.

IV. CONCLUSION

Assignment problem is an important structured linear programming problem. Classic assignment problem was the publication in 1955 of Kuhn's article on the Hungarian method for its solution. Over the past many years, many variations on the classic AP have been proposed. This article makes an effort to provide an overview of classic assignment problem.

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