On Various Cyclic Contractions In Dislocated Quasi B-Metric Spaces

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Abstract : In this paper we prove some new fixed point results in dislocated quasi b-metric spaces. We introduce dqb-cyclic-Chatterjea type contraction,dqb-cyclic-Ciric type contraction and dqb-cyclic-Reich type contraction. We also provide proofs for fixed point results using aforesaid contraction conditions.

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I. INTRODUCTION

Fixed Point Theory has witnessed enormous amount of research in past few decades. As Banach[3], in 1922, laid the foundations of contraction principle and devised the mechanism for ensuring and finding a fixed point to be existent provided the self-map be continuous, he opened up a broad opportunity to generalizations and applications of the contraction principle. The principle may be stated as follows:

"If X is a complete metric space, then every contraction mapping from X to itself has a unique fixed point."

In 1930, Wilson[10] introduced the concept of $quasi-metric$, which is merely suppressing one of the abiding axioms for being a complete metric i.e. $d(x,y)=d(y,x)$. Hitzler & Seda[6] introduced the concept of $dislocatedmetricspace$ and generalized the Banach's contraction principle. Ahmed, Hassan & Zeyada[12] introduced the concept of $dislocatedquasimetricspace$, which is an obvious generalization of $dislocatedmetricspace$ and Banach's contraction principle. In 1989, Bakhtin[2], introduced the concept of $b-metricspace$ and generalized Banach's contraction principle for $b-metricspace$. Chakkrid and Cholatis[8] introduced the concept of $dislocatedquasib-metricspace$.

In this paper we establish some new outcomes using Chatterjea contraction principle[4], Ciric contraction principle[5] and Rich contraction principle[9].

II. PRELIMINARIES

Definition 2.1[1]. Let X be a non empty set, let $d: X\times X\to[0,\infty)$ and let $k\in\mathbb{R}$.Then (X,d) is said to be $b-metricspace$if the following conditions are satisfied:

(i) $d(x,y)=0$ if and only if $x=y,\ \forall x,y\in X$.
(ii)$d(x,y)=d(y,x),\ \forall x,y\in X$.
(iii) There exist a real number$k\geq 1$ such that $d(x,y)\leq k[d(x,z)+d(z,y)], \forall x,y\in X$.

Definition 2.2[1]. Let Xbe a non empty set and let $d: X\times X\to\mathbb{R}$, then (X,d) is known as dislocated metric if the following conditions are met and$\forall x,y,z\in X$.

(i) $d(x,y)\geq 0,\ \forall x,y\in X$.
(ii)$d(x,y)=d(y,x),\ \forall x,y\in X$.
(iii) $d(x,y)=d(y,x)\Longrightarrow x=y,\ \forall x,y\in X$.
(iv)$d(x,y)\leq d(x,z)+d(z,y), \forall x,y\in X$.

If $d$satisfies (i),(iii) and (iv) then $d$is called $dislocatedquasimetric$on $X$and the pair $(X,d)$is called$dislocatedquasimetricspace$.

Definition 2.3[1]. Let X be a non-empty set and let the mapping $d: X\times X\to[0,\infty)$.Let a constant $k\geq 1$. Iffollowing conditions are satisfied:

(i) $d(x,y)=d(y,x)=0,\ \forall x,y\in X$.
(ii) $d(x,y)\leq k[d(x,z)+d(z,y)], \forall x,y\in X$.

Then the pair $(X,d)$is called $dislocatedquasib-metricspace$.

Definition 2.4[1]. Let (X,d)be a$dqb-metricspace$. A sequence $< x_n >$in X is called to be *dqb-converges* to $x\in X$ if

$$\lim_{n\to\infty} d(x_n,x)=0=\lim_{n\to\infty} d(x,x_n)$$

In this case x is called $dqb-limit$of $< x_n >$and is written as $x_n\to x$.

Definition 2.5[8]. Let (X, d) be a dqb-metric space. A sequence $< x_n >$in X is called as dqb-Cauchy sequence if

$$\lim_{n,m\to\infty} d(x_n,x_m)=0=\lim_{n\to\infty} d(x_m,x_n)$$

Proposition 2.6[8]. If (X,d) is a $dqb-metricspace$ then a function $f: X\to X$ is continuous if and only if $x_n\to x\Longrightarrow fx_n\to fx$.

Definition 2.7 Let A and B be nonempty closed subsets of a $metricspace$ (X,d) and $S: A\cup B \to A\cup B$. S is called a cyclic map iff $S(A)\subseteq B and S(B)\subseteq A$.

Definition 2.8[7]. A cyclic map $S: A\cup B \to A\cup B$ is said to be a cyclic contraction if there exists $\alpha\in[0,1)$ such that

$$d(Sx,Sy)\le \alpha d(x,y)$$

$\forall x\in A and y\in B$.

Definition2.9 Let A and B be non empty closed subsets of a $completed qb-metricspace$ (X,d). A cyclic mapping $S: A\cup B \to A\cup B$ is called a $dqb-cyclicChatterjeatypecontraction$ if there exist some $r\in(0,\frac{1}{2})$ such that

$$d(Sx,Sy)\le r[d(x,Sy)+d(y,Sx)]$$

$\forall$ x ∈ A, y ∈ B with $k\ge 1$ and $kr\le 1$.

Definition2.10 Let A and B be non empty closed subsets of a $completedqb-metricspace$ (X,d). A cyclic mapping $S: A\cup B \to A\cup B$ is called a $dqb-cyclicCirictypecontraction$ if there exist some $r\in(0,1)$ such that

$$d(Sx,Sy)\le r max[d(x,y),d(Sx,x),d(Sy,y)]$$

$\forall$ x ∈ A, y ∈ B with $k\ge 1$ and $kr\le 1$.

Definition2.11 Let A and B be non empty closed subsets of a $completedqb-metricspace$ (X,d). A cyclic mapping $S: A\cup B \to A\cup B$ is called a $dqb-cyclicReichtypecontraction$ if there exist some $a,b \& c where (a+b+c)<1$, such that

$$d(Sx,Sy)\le a.d(x,Sx)+b.d(y,Sy)+c.d(x,y)$$

$\forall x\in A, y\in B$ with $k\ge 1$ and $k(a+b+c)\le 1$.

III. MAIN RESULTS

Now, we prove $cyclic-Chatterjea\ type\ Contraction$ in $dislocated\ quasi\ b-metric\ space$.

Theorem 3.1 Let A and B be nonempty closed subsets of a $complete\ dislocated\ quasi-b-metric\ space$ (X,d). Let cyclic mapping $S: A\cup B\to A\cup B$ satisfies the condition of a $dqb-cyclic-Chatterjea\ type\ Contraction$. Then S has a unique fixed point in A ∩ B.

Proof: Let $x\in A(fix)$. Then, by the condition of the theorem,

$$d(S^2x,Sx)=d(S(Sx),Sx)$$
$$d(S^2x,Sx)\le r[d(Sx,Sx)+d(x,S^2x)]$$
$$d(S^2x,Sx)\le r\,\alpha \qquad (1)$$

And

$$d(Sx,S^2x)=d(Sx,S(Sx))$$
$$d(Sx,S^2x)\le r[d(x,S^2x)+d(Sx,Sx)]$$
$$d(Sx,S^2x)\le r[d(Sx,Sx)+d(x,S^2x)]$$
$$d(Sx,S^2x)\le r\,\alpha \qquad (2)$$

Where $\alpha=\left(d(Sx,Sx)+d(x,S^2x)\right)$

Now, from (1) and (2), we have, $d(S^3x,S^2x)\le r^2\alpha$, and $d(S^2x,S^3x)\le r^2\alpha$.

And in general $\forall\, n\in\mathbb{N}$, we get,

$$(S^{n+1}x,S^nx)\le r^n\alpha$$

And

$$(S^nx,S^{n+1}x)\le r^n\alpha$$

Let $n,m\in\mathbb{N}$ with $m>n$, by using the triangular inequality, we have

$$\begin{aligned} d(S^mx,S^nx) &\le k^{m-n}d(S^mx,S^{m-1}x)+k^{m-n-1}d(S^{m-1}x,S^{m-2}x)+\cdots+k(S^{n+1}x,S^nx)\\ &=(k^{m-n}r^{m-1}+k^{m-n-1}r^{m-2}++k^{m-n-2}r^{m-3}+\cdots+k^2r^{n+1}+kr^n)\alpha\\ &=((kr)^{m-n}r^{n-1}+(kr)^{m-n-1}r^{n-1}++(kr)^{m-n-2}r^{n-1}+\cdots+(kr)^2r^{n-1}+(kr)r^{n-1})\alpha\\ &\le(r^{n-1}+r^{n-1}+r^{n-1}+\cdots+r^{n-1}+r^{n-1})\alpha\\ &=(r^{n-1})(m-n-1)\alpha\\ d(S^mx,S^nx) &\le(r^{n-1})\beta\alpha \end{aligned}$$

With $\beta>0$, as $n\to\infty, we\ get\ d(S^mx,S^nx)\to 0$.

In a similar way, let $m,n\in\mathbb{N}, with\ m>n$, by using triangular inequality, we have

$$d(S^nx,S^mx)\le(r^{n-1})\beta\alpha$$

With $\beta>0$, as $n\to\infty, we\ get\ d(S^mx,S^nx)\to 0$. Therefore sequence $<S^nx>$ is a Cauchy sequence that converges to some $u\in X$. As (X,d) is complete, sequence $<S^nx>$ is in A and sequence $<S^{2n-1}x>$ is in B in a way that we have both the sequences tend to same limit $u\in X$.

Since $A\ and\ B$ are closed subsets of X and, $u\in A\cap B$, therefore $A\cap B\ne\emptyset$.

Now we will prove the existence of fixed point i.e. $Su=u$.

By the condition of the theorem we have

$$d(S^nx,Su)=d(S(S^{n-1}x),Su)$$
$$d(S^nx,Su)\le r[d(S^{n-1}x,Su)+d(u,S^nx)]$$

Now as $n\to\infty$, we get

$$d(u,Su)\le r[d(u,Su)+d(u,u)]$$
$$d(u,Su)\le r d(u,Su)$$

Since $r \geq 1$, this inequality is only possible if $d(u, Su) = 0$.
Similarly from the condition of the theorem, we have

$$d(Su, S^n x) = d\big(Su, S(S^{n-1}x)\big)$$
$$d(Su, S^n x) \leq r[d(u, S^n x) + d(S^{n-1}x, Su)]$$

Now as $n \to \infty$, we get

$$d(Su, u) \leq r[d(Su, u) + d(u, Su)]$$

From triangular inequality, $d(Su, u) \leq rd(Su, Su)$
Which gives $d(Su, u) = 0$.
Hence $d(u, Su) = d(Su, u) = 0$ and thus, $Su = u$. This implies that u is a fixed point of S.

Now we prove the uniqueness of the fixed point. Let $v \in X$ be another fixed point of S, such that $Sv = v$. Then by the condition of the theorem we lead to

$$d(u, v) = d(Su, Sv)$$
$$d(u, v) \leq r[d(u, Sv) + d(v, Su)]$$
$$d(u, v) \leq r[d(u, v) + d(v, u)]$$
$$d(u, v) = r\, d(u, u)$$
$$d(u, v) = 0.$$

Similarly,

$$d(v, u) = d(Sv, Su)$$
$$d(v, u) \leq r[d(v, Su) + d(u, Sv)]$$
$$d(v, u) \leq r[d(v, u) + d(u, v)]$$
$$d(v, u) = r\, d(v, v)$$
$$d(v, u) = 0.$$

$d(u, v) = d(v, u) = 0$. This implies that $u = v$ and u is the unique fixed point of S.
Now, we provide alternate proof for $cyclic - Ciric\ type\ Contraction$ in $dislocated\ quasi\ b - metric\ space$. Wu et.al.[8] in 2016 proved $cyclic - Ciric\ type\ Contraction$ in $dislocated\ quasi\ b - metric\ space$.
Theorem 3.2 Let A and B be nonempty closed subsets of a $complete\ dislocated\ quasi - b - metric\ space$ (X, d). Let cyclic mapping $S: A \cup B \to A \cup B$ satisfies the condition of a $dqb - cyclic - Ciric\ type\ Contraction$. Then S has a unique fixed point in A ∩ B.
Proof: Let $x \in A(fix)$.Then, by the condition of the theorem,

$$d(S^2x, Sx) = d(S(Sx), Sx)$$
$$d(S^2x, Sx) \leq r\, max[d(Sx, x) + d(S^2x, Sx) + d(Sx, x)]$$
$$\text{Let } \alpha = max[d(S^2x, Sx) + d(Sx, x)]$$
$$d(S^2x, Sx) \leq r\, \alpha \qquad (3)$$

And

$$d(Sx, S^2x) = d\big(Sx, S(Sx)\big)$$
$$d(Sx, S^2x) \leq r\, max[d(x, Sx) + d(Sx, x) + d(S^2x, Sx)]$$
$$\text{Let } \beta = max[d(x, Sx) + d(Sx, x) + d(S^2x, Sx)]$$
$$d(Sx, S^2x) \leq r\, \beta \qquad (4)$$

Let $\zeta = \max[\alpha, \beta]$
Now, from (3) and (4), we have, $d(S^3x, S^2x) \leq r^2\zeta$ and $d(S^2x, S^3x) \leq r^2\zeta$.
And more generally $\forall\, n \in \mathbb{N}$, we get,

$$(S^{n+1}x, S^n x) \leq r^n\zeta$$

And

$$(S^n x, S^{n+1}x) \leq r^n\zeta$$

Let $n, m \in \mathbb{N}$with $m > n$, by using the triangular inequality, we have

$$d(S^m x, S^n x) \leq k^{m-n} d(S^m x, S^{m-1}x) + k^{m-n-1} d(S^{m-1}x, S^{m-2}x) + \cdots + k(S^{n+1}x, S^n x)$$
$$= (k^{m-n}r^{m-1} + k^{m-n-1}r^{m-2} + +k^{m-n-2}r^{m-3} + \cdots + k^2r^{n+1} + kr^n)\zeta$$
$$= ((kr)^{m-n}r^{n-1} + (kr)^{m-n-1}r^{n-1} + +(kr)^{m-n-2}r^{n-1} + \cdots + (kr)^2r^{n-1} + (kr)r^{n-1})\zeta$$
$$\leq (r^{n-1} + r^{n-1} + r^{n-1} + \cdots + r^{n-1} + r^{n-1})\zeta$$
$$= (r^{n-1})(m - n - 1)\zeta$$
$$d(S^m x, S^n x) \leq (r^{n-1})\lambda\zeta$$

With $\lambda > 0$, as $n \to \infty, we\ get\ d(S^m x, S^n x) \to 0$.
In a similar way, let $m, n \in \mathbb{N}, with\ m > n$, by using triangular inequality, we have

$$d(S^n x, S^m x) \leq (r^{n-1})\lambda\zeta$$

With $\lambda > 0$, as $n \to \infty, we\ get\ d(S^m x, S^n x) \to 0$. Therefore sequence $< S^n x >$ is a Cauchy sequence that converges to some $u \in X$. As (X, d) is complete, sequence $< S^n x >$is in A and sequence $< S^{2n-1}x >$ is in B in a way that we have both the sequences tend to same limit $u \in X$.
Since $A\ and\ B$ are closed subsets of X and, $u \in A \cap B$, therefore $A \cap B \neq \emptyset$.
Now we will prove the existence of fixed point i.e. $Su = u$.
By the condition of the theorem we have

$$d(S^n x, Su) = d(S(S^{n-1}x), Su)$$
$$d(S^n x, Su) \leq r\, max[d(S^{n-1}x, u) + d(S^n x, S^{n-1}x) + d(Su, u)]$$

Now as $n \to \infty$, we get

$$d(u, Su) \leq r\, max[d(u, Su) + d(u, u) + d(Su, Su)]$$
$$d(u, Su) \leq r d(u, Su)$$

Since $r \geq 1$, this inequality is only possible if $d(u, Su) = 0$.
Similarly from the condition of the theorem, we have

$$d(Su, S^n x) = d\big(Su, S(S^{n-1}x)\big)$$
$$d(Su, S^n x) \leq r\, max[d(u, S^{n-1}x) + d(Su, u) + d(S^n x, S^{n-1}x)]$$

Now as $n \to \infty$, we get

$$d(Su, u) \leq r\, max[d(u, u) + d(Su, Su) + d(u, u)]$$

Which gives $d(Su, u) = 0$.
Hence $d(u, Su) = d(Su, u) = 0$ and thus, $Su = u$. This implies that u is a fixed point of S.

Now we prove the uniqueness of the fixed point. Let $v \in X$ be another fixed point of S, such that $Sv = v$. Then by the condition of the theorem we lead to

$$d(u, v) = d(Su, Sv)$$
$$d(u, v) \leq r\, max[d(u, v) + d(Su, u) + d(Sv, v)]$$
$$d(u, v) \leq r\, max[d(u, v) + d(u, u) + d(v, v)]$$
$$d(u, v) \leq r[d(u, v)]$$
$$d(u, v) = 0.$$

Similarly,

$$d(v, u) = d(Sv, Su)$$
$$d(v, u) \leq r\, max\, [d(v, u) + d(Sv, v) + d(Su, u)]$$
$$d(v, u) \leq r\, max[d(v, u) + d(v, v) + d(u, u)]$$
$$d(v, u) \leq r[d(v, u)]$$
$$d(v, u) = 0.$$

$d(u, v) = d(v, u) = 0$. This implies that $u = v$ and u is the unique fixed point of S.
Now, we prove $cyclic - Reich\ type\ Contraction$ in $dislocated\ quasi\ b - metric\ space$.

Theorem 3.3 Let A and B be nonempty closed subsets of a $complete\ dislocated\ quasi - b - metric\ space$ (X, d). Let cyclic mapping $S: A \cup B \to A \cup B$ satisfies the condition of a $dqb - cyclic - Reich\ type\ Contraction$. Then S has a unique fixed point in A ∩ B.

Proof: *Let* x be some arbitrary in X. We define a sequence $< x_n >$ in X such that $x_1 = S(x_0)$, $x_2 = S(x_1)$ ... in general $S(x_{2n}) = x_{2n+1}$, $S(x_{2n+1}) = x_{2n+2}\ for\ n = 0, 1, 2, 3$ ...Then, by the condition of the theorem,

$$d(x_1, x_2) = d(Sx_0, Sx_1)$$
$$d(x_1, x_2) \leq a.d(x_0, Sx_0) + b.d(x_1, Sx_1) + c.d(x_0, x_1)$$
$$d(x_1, x_2) \leq a.d(x_0, x_1) + b.d(x_1, x_2) + c.d(x_0, x_1)$$
$$d(x_1, x_2) \leq (a + c)d(x_0, x_1) + b.d(x_1, x_2)$$
$$d(x_1, x_2) - b.d(x_1, x_2) \leq (a + c)d(x_0, x_1)$$
$$(1 - b)d(x_1, x_2) \leq (a + c)d(x_0, x_1)$$
$$d(x_1, x_2) \leq \left(\frac{a + c}{1 - b}\right) d(x_0, x_1)$$

In a similar way we have, $d(x_2, x_3) \leq \left(\frac{a+c}{1-b}\right) d(x_1, x_2)$

or, $d(x_2, x_3) \leq \left(\frac{a+c}{1-b}\right)^2 d(x_0, x_1)$

And continuing like this we have, $d(x_n, x_{n+1}) \leq \left(\frac{a+c}{1-b}\right)^n d(x_0, x_1)$

$$d(x_{n+1}, x_{n+2}) \leq \left(\frac{a + c}{1 - b}\right)^{n+1} d(x_0, x_1)$$

Now as $n \to \infty$, $\left(\frac{a+c}{1-b}\right)^{n+1} d(x_0, x_1) \to 0$, This, therefore, suggests that the sequence $\{x_n\}\ is\ a\ Cauchy\ Sequence$ in X.Thus, there is a point $u \in X$ such that $x_n \to u$. Therefore we have, $Su = u$. Now we prove the uniqueness of the fixed point. Let $v \in X$ be another fixed point of S, such that $Sv = v$. Then by the condition of the theorem we have

$$d(u, v) = d(Su, Sv)$$
$$d(u, v) \leq a.d(u, Su) + b.d(v, Sv) + c.d(u, v)$$
$$d(u, v) \leq a.d(u, u) + b.d(v, v) + c.d(u, v)$$
$$d(u, v) \leq c.d(u, v)$$
$$d(u, v) = 0.$$

Similarly,

$$d(v, u) = d(Sv, Su)$$
$$d(v, u) \leq a.d(v, Sv) + b.d(u, Su) + c.d(v, u)$$

$$d(v,u) \le a.d(v,v) + b.d(u,u) + c.d(v,u)$$
$$d(v,u) \le c.d(v,u)$$
$$d(v,u) = 0.$$

$d(u,v) = d(v,u) = 0$.This implies that $u = v$ and u is the unique fixed point of S.

IV. CONCLUSION

We presented some new results and provided alternate proof for fixed point result in *dislocated quasi b-metric space* using *dqb-cyclic Ciric type contraction.*

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