ON THE CUBIC EQUATION $x^3 + y^3 + 6(x+y)z^2 = 4w^3$

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Abstract:
An attempt is made to solve the cubic equation with four unknowns given by $x^3 + y^3 + 6(x+y)z^2 = 4w^3$ in integers. Some special relations between the solutions are given.

Keywords: Homogeneous cubic, cubic with four unknowns, integer solutions.

I. Introduction

It is well-known that there are varieties of cubic equations with four unknowns to obtain integer solutions satisfying them. In particular, different choices of cubic equations with four unknowns are presented in [1-6]. This paper has a different choice of cubic equation with four unknowns given by $x^3 + y^3 + 6(x+y)z^2 = 4w^3$ to obtain a sequence of integer solutions. Special connections between the solutions are exhibited.

II. Notations

- $GNO_n = 2n-1$
- $PR_n = n(n+1)$
- $t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$
- $S_n = 6n^2 - 6n + 1$

III. Method of Analysis

The cubic equation with four unknowns to be solved is

$$x^3 + y^3 + 6(x+y)z^2 = 4w^3 \tag{1}$$

Introducing the linear transformations

$$x = u+v,\ y = u-v,\ w = u \tag{2}$$

in (1), it is written as

$$u^2 = 3v^2 + 6z^2 \tag{3}$$

Again, the substitution of $u = 3U,\ v = X + 6T,\ z = X - 3T$ (4)

in (3), leads to

$$U^2 = 18T^2 + X^2 \tag{5}$$

whose solutions are

$$T = 2rs,\ X = 18r^2 - s^2,\ U = 18r^2 + s^2 \tag{6}$$

From (6),(4), (2), the solutions of (1) are given below:

$$x = x(r,s) = 72r^2 + 2s^2 + 12rs$$

$$y = y(r,s) = 36r^2 + 4s^2 - 12rs$$

$$z = z(r,s) = 18r^2 - s^2 - 6rs$$

$$w = w(r,s) = 54r^2 + 3s^2$$

Properties:

- $x(1,s)-2y(1,s)+6PR_s-21GNO_s-21=0$
- $y(r,s)-2z(r,s)$ is a Nasty number
- $x(r,1)-4z(r,1)-18GNO_r\equiv 0(\bmod 2)$
- $w(r,1)-9S_r-27GNO_r=21$
- $z(r,1)-11r+1=t_{38,r}$

Remark: One may also consider the transformation (4) as

$$u=3U,\ v=X-6T,\ z=X+3T \tag{7}$$

In this case, the corresponding values of x,y,z,w satisfying (1) are represented by

$$x=x(r,s)=72r^2+2s^2-12rs$$

$$y=y(r,s)=36r^2+4s^2+12rs$$

$$z=z(r,s)=18r^2-s^2+6rs$$

$$w=w(r,s)=54r^2+3s^2$$

In addition to the above solutions, other sets of solutions to (1) may be obtained as illustrated below:
Note that (5) is represented as the pair of equations as in Table 1:

Table 1: Pair of equations

S.No	**1**	**2**	**3**	**4**	**5**	**6**
$U+X$	T^2	$9T^2$	$3T^2$	$9T$	$6T$	$18T$
$U-X$	18	2	6	$2T$	$3T$	T

Substituting the corresponding values of U,X and T from the above Table1 and in (4) and (2) the different sets of integer solutions to (1) thus obtained are shown below in Table 2:

Table 2: Solutions

S.No	***x***	***y***	***z***	***w***
1	$8k^2+12k+18$	$4k^2-12k+36$	$2k^2-6k-9$	$6k^2+27$
2	$72k^2+12k+2$	$36k^2-12k+4$	$18k^2-6k-1$	$54k^2+3$
3	$24k^2+12k+6$	$12k^2-12k+12$	$6k^2-6k-3$	$18k^2+9$
4	$54k$	$14k$	k	$33k$
5	$42k$	$12k$	$-3k$	$27k$
6	$86k$	$28k$	$11k$	$57k$

Note: Substituting the values of U,X,T obtained from Table 1 in (7) and (2), one obtains some more choices of solutions to (1).
Also, (5) is taken as

$$X^2+18T^2=U^2*1 \tag{8}$$

Consider

$$U=a^2+18b^2 \tag{9}$$

and

$$1=\frac{(3+i\,2\sqrt{18})(3-i\,2\sqrt{18})}{81} \tag{10}$$

Applying (9),(10) in (8) and factorizing, take

$$X+i\sqrt{18}T=\frac{(a+i\sqrt{18}b)^2(3+i\,2\sqrt{18})}{9}$$

from which note that

$$X=\frac{1}{9}(3a^2-54b^2-72ab)$$

$$T=\frac{1}{9}(2a^2-36b^2+6ab)$$

Replacing a by $3A$ in the above equations, we have

$$\left.\begin{aligned} X &= 3A^2 - 6b^2 - 24Ab \\ T &= 2A^2 - 4b^2 + 2Ab \end{aligned}\right\} \quad (11)$$

and from (9),

$$U = 9A^2 + 18b^2 \quad (12)$$

Substituting (11) and (12) in (4) and (2), the corresponding values of x, y, z, w satisfying (1) are given by

$$x = 42A^2 + 24b^2 - 12Ab$$

$$y = 12A^2 + 84b^2 + 12Ab$$

$$z = -3A^2 + 6b^2 - 30Ab$$

$$w = 27A^2 + 54b^2$$

Note: In addition to (10), one may also write 1 as

$$1 = \frac{\left(7 + i\,2\sqrt{18}\right)\left(7 - i\,2\sqrt{18}\right)}{121}$$

The repetition of the above process leads to a different set of solutions to (1).

In this paper, an attempt has been made to obtain different sets of integer solutions to $x^3 + y^3 + 6(x+y)z^2 = 4w^3$. In conclusion, a search for determining integer solutions to the considered cubic equation with four unknowns may be performed.

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