Hadamard Matrix and its Application in Combinatorial Design Theory

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Abstract : Matrix Theory is Widely used in a variety of areas including applied mathematics, computer science, economics, engineering, operations Research, statistics and others. This paper presents the study of Hadamard matrices.There are theorems and properties related to this matrix. Also the construction of Hadamard 2-Design using hadamard matrices is discussed in this paper.

Keywords: Orthogonal, eigen values ,Tensor product.

I. INTRODUCTION

The origin of mathematical matrices lie with the study of systems of simultaneous linear equations ,but they were known as arrays until the 1800s.The term "matrix" was introduced by James Joseph Sylvester, an English mathematician in the year 1850.

Matrices find many applications in scientific fields and apply to practical real life problems as well, thus making an indispensable concept for solving many practical problems. In computer based applications, matrices play a vital role in the projection of three dimensional image into a two dimensional screen, creating the realistic seeming motions. This paper studies a special type of matrix namely Hadamard matrix and also discusses the application of hadamard matrix in combinatorial design theory.

II. PRELIMINARIES

Orthogonal vectors: Two vectors x and y, in an inner product space v, are orthogonal if their inner product <x,y> is zero.

Orthogonal set: A set $\{v_1, v_2, \ldots, v_n\}$ is said to be orthogonal, if $<v_i, v_j>$ is zero for $i \neq j$.

Norm of a vector: Norm of a vector x is given as $\|x\| = (x,x)^{1/2}$.

Identity matrix I_n: An n x n matrix I_n such that $AI_n = A$ where A is an n x n matrix ,is called an identity matrix.

Orthogonal matrix: An n x n matrix A is orthogonal if $AA^T = A^TA = I_n$.
det(A): Determinant of a matrix A.
A^T: transpose of a matrix.

Eigen values: A real number λ is said to be an eigen value of an n x n matrix A, if there exists a non-zero vector $x \epsilon R^n$ such that $Ax = \lambda x$

Tensor Product of two matrices :

Tensor product of two matrices A and B where

$$A = \begin{bmatrix} a11 & \cdots & a1n \\ \vdots & \ddots & \vdots \\ am1 & \cdots & amn \end{bmatrix}$$ and

$$B = \begin{bmatrix} b11 & \cdots & b1q \\ \vdots & \ddots & \vdots \\ bm1 & \cdots & bpq \end{bmatrix}$$ is an mp x nq matrix defined as

$$A \times B = \begin{bmatrix} a11B & \cdots & a1nB \\ \vdots & \ddots & \vdots \\ am1B & \cdots & amnB \end{bmatrix}$$

III. HADAMARD MATRICES

Definition:

An n-Square matrix H with entries + 1 and - 1 such that the set of its row vectors(or Column vectors) forms an orthogonal set is called a Hadamard matrix.

For example,

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_4 = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{matrix}$$

are hadamard matrices.

Theorems and properties of Hadamard matrices

1.If A is an n-square Hadamard matrix ,then $H = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$ is also Hadamard.

Proof: since A is Hadamard , we have $AA^T= n I_n$.

Consider,

$$HH^T= \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$$

$$= \begin{pmatrix} 2nIn & 0 \\ 0 & 2nIn \end{pmatrix} =2n I_{2n}$$

$HH^T=2n I_{2n}$. since A is Hadamard ,its entries are + 1 and – 1. Hence the entries of H are +1 and -1.
$HH^T=m I_m$ where m=2n is the order of H and Hence it is Hadamard.

2.If H is an n-square Hadamard matrix, then $H / \sqrt{n}$ is orthogonal.

Proof: since H is an n-Square Hadamard matrix,

$HH^T=nI_n$

$(H/\sqrt{n})(H/\sqrt{n})^T=I_n$

$H/\sqrt{n}$ is orthogonal.

3. For an n-square Hadamard matrix H , $\|Hx\|=\sqrt{n}\ \|x\|$

Proof: consider $\|Hx\|^2= <Hx ,Hx>$
$= <x, H^T Hx>$
$=< x, n I_n x >$
$= n \|x\|^2$

Hence $\|Hx\|=\sqrt{n}\ \|x\|$.

4.$\det(H)= \pm(n)^{n/2}$ where H is an n-square Hadamard matrix.

Proof: Since H is Hadamard $HH^T=n I_n$

$\det(HH^T)= \det(n I_n)$
$=n^n \det(I_n)$
$|\det(H)|^2 =(n)^n$
$\det(H)=\pm (n)^{n/2}$

5.H_1 and H_2 are Hadamard matrices , then the tensor product $H_1 x H_2$ is also hadamard [5].

Proof: let H_1 be an n-square Hadamard matrix and H_2be an m-square Hadamard matrix.

Then $H_1H_1^T=n I_n$ and $H_2H_2^T =m I_m$.

$(H_1xH_2)(H_1xH_2)^T= (H_1H_1^T)(H_2H_2^T)$
$=(nIn)(mI_m)$
$=(mn)I_{mn}$

$H_1x H_2$ is an mn –square matrix.

Hence $H_1x H_2$ is Hadamard.

IV. HADAMARD 2-DESIGNS.

Combinatorial design theory is the part of combinatorial mathematics that deals with the existence ,construction and properties of systems of finite sets whose arrangement satisfy generalized concepts of symmetry. Modern application of combinatorial design theory are found in a wide range of areas including finite geometry, tournament scheduling, lotteries, mathematical biology ,networking and cryptography.

Definitions [4]

1)2-design: let S={1,2,3,…..n} be a set of n elements. A collection ζ of distinct subsets of S is called (n,k, λ) 2-design if

i)2≤ k<n.

ii) $\lambda > 0$

iii) each set in ζ contains exactly k- elements.

iv) each 2- element subset of S is contained in exactly λ of the sets in ζ .

For example,

Let n=7 and S={1,2,3,4,5,6,7}

Then the sets {1,2,4},{2,3,5},{3,4,6},{4,5,7},{5,6,1},{6,7,2},{7,1,3} forms a (7,3,1) 2-design.

2) Hadamard 2-design:

A (4n-1,2n-1,n-1) 2-design is called a Hadamard 2-design of order n.

For example,

Let n=7 and S={1,2,3,4,5,6,7}

Then the sets {1,2,4},{2,3,5},{3,4,6},{4,5,7},{5,6,1},{6,7,2},{7,1,3} forms a (7,3,1) Hadamard 2-designs of order 2

V. CONCLUSION

In this paper we studies Hadamard matrix and its properties. It is concluded that absolute value of determinant and eigen values of n-square Hadamard matrix is $n^{n/2}$ and √n respectively. The application of Hadamard matrix in combinatorial Design theory is included.

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