"The study of harmonic oscillator and conventional coherent states using supersymmetric quantum mechanical approach "

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Abstract

Supersymmetric quantum mechanics has been developed as an elegant analytical approach to one dimensional problems. It generalizes the ladder operator approach used in the study of the harmonic oscillator. In this treatment, the factorization of a one dimensional Hamiltonian obtained using " charge operators". For 1D harmonic oscillator, lowering and raising charge operators can be used. It not only allow the factorization of 1D Hamiltonian but also form Lie algebraic structure which generates isospectral SUSY partner Hamiltonians. In addition several different approaches have been employed to study generalized and approximate coherent states of systems other than harmonic oscillator. In this paper, algebraic treatments being applied to the extension of coherent states for shape-invariant systems.

Key words: Harmonic oscillator, Conventional coherent states and charge operators.

1. Introduction:

The eigenstates of the various partner Hamiltonians are connected by applications of the charge operators. As an analytical approach, SUSY- QM approach has been utilized to study a number of quantum mechanical problems including the Morse oscillator [6] and the radial hydrogen atom equation. It can be used in discovery of new exactly solvable potentials. The harmonic oscillator is fundamental to a wide range of physics including the electromagnetic field, spectroscopy, solid state physics, coherent state theory and SUSYQM. The broad application of the harmonic oscillator stems from lowering and raising ladder operators which can be used to factor the

Hamiltonian of the system. For example, cannonical coherent states are defined as the eigenstates of the lowering operator of the harmonic oscillator and they are also minimum uncertainty states which minimize the Heisenberg uncertainty product for position and momentum. The lowering operator of the harmonic oscillator annihilates the ground states and it minimizes the HUP (Heisenberg Uncertainty Product).

Conventional harmonic oscillator coherent states correspond to those states which minimizes the position – momentum uncertainty relation. However, these harmonic oscillator coherent states are also constructed by applying shift operators labelled with points of the discrete phase space to the ground state of the harmonic oscillator, known as " fiducial state "[7]. But, Klauder and Skagerstam choose to define coherent states in broadest sense. Similarly, the charge operator in SUSY-QM annihilates the ground state of the corresponding system. Let us construct system-specific coherent states for any bound quantum system by similarity between treatment of harmonic oscillator and supersymmetric quantum mechanics

2. <u>HARMONIC OSCILLATOR AND CONVENTIONAL COHERENT STATES</u>:

The Hamiltonian of the harmonic oscillator can be expressed as

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2, \qquad .....(1)$$

where

m=mass of the particle,

$and\ \omega$= angular frequency of the oscillator.

The Hamiltonian of system in terms of the raising and lowering operators is given by

$$H = \hbar\omega\left(\bar{a}^+\bar{a} + \frac{1}{2}\right), \qquad .....(2)$$

Where $\bar{a}^+$ is the raising operator and $\bar{a}$ is the lowering operator.

These two operators can be expressed in terms of the position operator $\bar{x}$ and corresponding momentum operator $\bar{p}_x$ as

$$\bar{a} = \sqrt{\frac{m\omega}{2\hbar}}\bar{x} + \frac{i\bar{p}_x}{\sqrt{2m\hbar\omega}}, \qquad ....(3)$$

$$and\ \ \bar{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\bar{x} - \frac{i\bar{p}_x}{\sqrt{2m\hbar\omega}}, \qquad ....(4)$$

For convenience, we take $\hbar = 2m = 1$ and $\omega = 2$.

In particular, the ground state of the harmonic oscillator is annihilated by the lowering operator

$$\bar{a}\psi_0 = \frac{1}{\sqrt{2}}(\bar{x} + i\bar{p}_x)\psi_0 = 0 \qquad .....(5)$$

On solving equation(5), we obtained the ground state wave function as

$$\psi_0(x) = \langle x|0\rangle = Ne^{-x^2/2}, \qquad .....(6)$$

where N is the normalization constant.

One of the important properties for the ground state of the harmonic oscillator is that the ground state is a minimum uncertainty state, which minimizes the Heisenberg uncertainty product $\Delta\bar{x}\Delta\bar{p}_x$. Using Schwarz's inequality, HU principle gives

$$\langle\psi|\bar{x}^2|\psi\rangle\langle\psi|\bar{p}_x^2|\psi\rangle \geq |\langle\psi|\bar{x}\bar{p}_x|\psi\rangle|^2, \qquad .....(7)$$

Assuming zero expectation values of the position and momentum operators for convenience. The equality holds for the state $|\psi\rangle$ such that

$$\bar{x}|\psi\rangle = -i\sigma^2\bar{p}_x|\psi\rangle \qquad ....(8)$$

where σ^2 is real and greater than zero.

As noted in equation(5), the ground state of the harmonic oscillator satisfies the relation with $a^2 = 1$, and hence it is a minimum uncertainty state. In fact, the ground state corresponds to a state centred in the phase space at $x = 0$ and $k = 0$.

Harmonic oscillator coherent states can be constructed by using shift operators labeled with points of the discrete phase space to a fiducial state, which is taken as the ground state of the harmonic oscillator ([5,10]). In this case, harmonic oscillator coherent states are generated by $|\propto\rangle = \bar{D}(\propto)|0\rangle$. The shift operator is given by

$$\bar{D}(\propto) = e^{\propto\bar{a}^+ - \propto^+\bar{a}} \qquad .....(9)$$

Where $$\propto = \frac{1}{\sqrt{2}}\left[\frac{x}{\sigma} + ik\sigma\right] \qquad .....(10)$$

Here $\propto$ is a complex-number representation of the phase point x and k and the quantity σ is a scaling parameter with the dimensions of length. Thus, the harmonic oscillator coherent states can be constructed by applying the shift operator to the ground state of the harmonic oscillator.

3. SUPERSYMMETRIC QUANTUM MECHANICS:

For one-dimensional SUSY-QM, the superpotential W is defined in terms of the ground state wave function by the Riccati substitution as

$$\psi_0^{(1)}(x) = N exp\left[-\int_0^x W_1(x')dx'\right], \qquad \ldots\ldots..(11)$$

Where N is the normalization constant.

The index [1] indicates that the ground state wave function and the superpotential are associated with the sector one Hamiltonian. It is assumed that Equation(11) solves the Schrodinger equationwith energy equal to zero such that

$$-\frac{d^2\psi_0^{(1)}}{dx^2} + V_1\psi_0^{(1)} = 0 \qquad \ldots\ldots.(12)$$

This does not impose any restriction since the energy can be changed by adding any constant to the Hamiltonian. From Equation(11), the superpotential can be expressed in terms of the ground state wave function as

$$W_1(x) = -\frac{d}{dx}\, ln\, \psi_0^{(1)}(x) \qquad \ldots..(13)$$

Substituting equation(11) into the Schroedinger equation (12), then we have Riccati equationfor the superpotential as

$$\frac{dW_1(x)}{dx} - W_1^2(x) + V_1(x) = 0 \qquad \ldots\ldots(14)$$

On the other hand, if $W_1(x)$ is known, then $V_1(x)$ is given by

$$V_1(x) = \left(W_1^2(x) - \frac{dW_1(x)}{dx}\right) \qquad \ldots\ldots(15)$$

Obviously, the Schroedinger equationin Equation(12) is equivalent to

$$-\frac{d^2\psi_0^{(1)}}{dx^2} + \left(W_1^2 - \frac{dW_1}{dx}\right)\psi_0^{(1)} = 0 \qquad ..(16)$$

Similar to the harmonic oscillator, the Hamiltonian operator can be factorized by introducing the " charge " operator and its adjoint, then

$$Q_1 = \frac{d}{dx} + W_1 = W_1 + i\bar{p}_x, \qquad \ldots\ldots(17)$$

$$Q_1^+ = -\frac{d}{dx} + W_1 = W_1 - i\bar{p}_x, \qquad \ldots\ldots(18)$$

where $\bar{p}_x = -i(d/dx)$ is the coordinate representation of the momentum operator. The ground state wave function $\psi_0(x)$ is assumed to be purely real; hence, the superpotential $W(x)$ is self-adjoint. Then, the sector one Hamiltonian is defined as $H_1 = Q_1^+ Q_1$.

Since $E_0^{(1)} = 0$ for $n = 0$, it follows from the Schroedinger equationthat for $n > 0$, we have

$$Q_1^+ Q_1 \psi_n^{(1)} = E_n^{(1)} = E_n^{(1)} \psi_n^{(1)} \qquad \text{......(19)}$$

where $\psi_n^{(1)}$ is an eigenstate of H_1 with $E_n^{(1)} \neq 0$.

Applying Q_1 to this equation, we obtain

$$H_2\left(Q_1\psi_n^{(1)}\right) = Q_1 Q_1^+ \left(Q_1\psi_n^{(1)}\right) = E_n^{(1)}\left(Q_1\psi_n^{(1)}\right) \quad \text{... ....(20)}$$

where the sector two Hamiltonian is defined as $H_2 = Q_1 Q_1^+$. Thus, $Q_1\psi_n^{(1)}$ is an eigenstate of H_2 with same energy $E_n^{(1)}$ as the state $\psi_n^{(1)}$. Similarly, let us consider the eigenstates of H_2 as

$$H_2\psi_n^{(2)} = Q_1 Q_1^+ \psi_n^{(2)} = E_n^{(2)} \psi_n^{(2)} \qquad \text{.......(21)}$$

Applying Q_1^+ to this equation, we obtained that $Q_1^+\psi_n^{(2)}$ is an eigenstate of H_1 then

$$H_1\left(Q_1^+\psi_n^{(2)}\right) = (Q_1^+ Q_1)\left(Q_1^+\psi_n^{(2)}\right) = E_n^{(2)}(Q_1^+\psi_n^{(2)} \qquad \text{.........(22)}$$

It follows that the Hamiltonians H_1 *and* H_2 have identical spectra with the exception of the ground state with $E_0^{(1)} = 0$. For the ground state, $Q_1\psi_0^{(1)} = 0$, and this shows that the quantity $Q_1\psi_0^{(1)}$ cannot be used to generate the ground state of the sector two Hamiltonian. Because of the uniqueness of the ground state with $E_0^{(1)} = 0$, the indexing of the first and second sector levels must be modified. It is clear that the eigenvalues and eigenfunctions of the two Hamiltonians H_1 *and* H_2 are related as

$$E_n^{(2)} = E_{n+1}^{(1)}, E_0^{(1)} = 0, \qquad \text{....(23)}$$

and $$\psi_n^{(2)} = \frac{Q_1\psi_{n+1}^{(1)}}{\sqrt{E_{n+1}^{(1)}}}, \psi_{n+1}^{(1)} = \frac{Q_1^+\psi_n^{(2)}}{\sqrt{E_n^{(2)}}} \qquad \text{.....(24)}$$

Similarly, starting from H_2 whose ground state energy is $E_0^{(2)} = E_1^{(1)}$, we can generate the sector three Hamiltonian H_3 as SUSY partner of H_2. This procedure can be continued until the number of bound excited states supported by H_1 is exhausted.

4. Results and Discussion:

The application of SUSY-QM to non relativistic quantum systems generalizes the powerful ladder operator approach used in the treatment of the harmonic oscillator. The lowering operator of the harmonic oscillator annihilates the ground state, while the charge operator annihilates the ground state of corresponding quantum systems. The similarity between the lowering operator of harmonic oscillator and SUSY charge operator implies that the superpotential can be regarded as a system-specific generalized displacement variable. Analogous to the ground state of the harmonic oscillator which minimizes the HUP, the ground state of any bound quantum system was identified as minimizer of SUSY HUP. It was observed that such dynamically adopted coherent states yields significantly more accurate excited state energies and wave functions than were obtained with the same number of the conventional coherent states and from the standard harmonic oscillator basis.

5. Conclusion:

The ladder operator approach of the harmonic oscillator and SUSY-QM formulation share strong similarity. This observation suggests that connection of the SUSY-QM with Heisenberg minimum uncertainty (μ^-) wavelets should be explored. The SUSY-displacement with the SUSY HUP can lead to the construction of the SUSY minimum uncertainty wavelets and the SUSY distributed approximating functions. These new functions and their potential applications in mathematics and physics are in progress.

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